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Number Theory and Discrete Mathematics

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Preface

To mark the World Mathematical Year 2000 an International Conference on Number Theory and Discrete Mathematics in honour of the legendary Indian Mathematician Srinivasa Ramanujan was held at the centre for Advanced study in Mathematics, Panjab University, Chandigarh, India during October 2–6, 2000. This volume contains the proceedings of that conference. In all there were 82 participants including 14 overseas participants from Austria, France, Hungary, Italy, Japan, Korea, Singapore and the USA. The conference was inaugurated by Prof. K.N. Pathak, Hon. Vice-Chancellor, Panjab University, Chandigarh on October 2, 2000. Prof. Bruce C. Berndt of the University of Illinois, Urbana – Chaimpaign, USA delivered the key note address entitled "The Life, Notebooks and Mathematical Contributions of Srinivasa Ramanujan". He described Ramanujan ~~as~~ one of this century's most influential Mathematicians. Quoting Mark Kac, Prof. George E. Andrews of the Pennsylvania State University, USA, in his message for the conference, described Ramanujan as a "magical genius".

During the 5-day deliberations invited speakers gave talks on various topics in number theory and discrete mathematics. We mention here a few of them just as a sampling:

- **M. Waldschmidt**, in his article, provides a very nice introduction to the topic of multiple polylogarithms and their special values.
- **C. Krattenthaler** studies the probability that a random tiling by rhombi of a hexagon with side lengths $2n + a$, $2n + b$, $2n + c$, $2n + a$, $2n + b$, $2n + c$ contains the (horizontal) rhombus with coordinates $(2n + x, 2n + y)$.
- **I. Katai** provides an uptodate survey of results concerning q-additive and q-multiplicative functions which are defined either on the set of natural numbers or on the set of prime numbers.
- **M. Elia** gives a nice collection of his observations about two equations introduced by Ramanujan.
- **B.C. Berndt** has contributed two articles. In one of them he discusses the influence of Carr's synopsis on Ramanujan while the second is devoted to Ramanujan's solutions to questions posed in the Journal of the Indian Mathematical Society, Vol. 4, 1912.
- **Kanemitsu, Tanigawa** and **Yoshimato** give a good account of some Ramanujan-type rapidly convergent series for special values of Dirichlet L-functions.
- **Padmavathamma** and **R. Salestina** make a substantial contribution to a 25-year old Andrews' conjecture about a partition identity.

- **P.D. Chawathe** and **V. Krishna** prove that all complete m-ary trees have antimagic labelings for $m \geq 2$.
- **S.S. Sane** provides a broad survey on the recent developments in symmetric quasi-symmetric and quasi-multiple designs.

In view of the feedback received from the participants we can say that the conference was very stimulating and indeed achieved its goals. The organization of the conference involved the efforts of many. On behalf of the organizing committee and on my own behalf, I record my thanks to Prof. K.N. Pathak, Hon. Vice-Chancellor, Panjab University, Chandigarh for his support, encouragement and for making the excellent facilities of the university available to us for the organization of the conference. Our sincere thanks are due to Professors G.E. Andrews, R.P. Bambah, B.C. Berndt, C. Krattenthaler, G.L. Mullen, V.C. Nanda, Padmavathamma, K. Ramachandra, K. Srinivasa Rao, M.V. Subbarao, M. Waldschmidt, those experts who have helped us in refereeing the articles but preferred to remain anonymous, all the invited speakers, my colleagues at Panjab University, particularly, Professors R.J. Hans-Gill, M. Raka, D.S. Khassa and V.K. Grover and my students Amarinder Singh, Amrit Pal Singh, Rachna, Sushma and Vandana.

Finally, I thank Mr. J.K. Jain, Partner, Hindustan Book Agency and his staff for their continued interest in this volume and for their proficient handling of its publication. In spite of all checks imposed to ensure accuracy, it is not reasonable to believe that the volume is error-free. I would appreciate receiving from readers any criticisms of the material and the identification of any errors.

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Multiple Polylogarithms: An Introduction

M. Waldschmidt

Dedicated to Professor R.P. Bambah on his 75th birthday

Multiple polylogarithms in a single variable are defined by

$$\mathrm{Li}_{(s_1,\dots,s_k)}(z) = \sum_{n_1>n_2>\cdots>n_k\geq 1} \frac{z^{n_1}}{n_1^{s_1}\cdots n_k^{s_k}},$$

when $s_1, \dots, s_k$ are positive integers and z a complex number in the unit disk. For $k = 1$, this is the classical polylogarithm $\mathrm{Li}_s(z)$. These multiple polylogarithms can be defined also in terms of iterated Chen integrals and satisfy *shuffle relations*. Multiple polylogarithms in several variables are defined for $s_i \geq 1$ and $|z_i| < 1 (1 \leq i \leq k)$ by

$$\mathrm{Li}_{(s_1,\dots,s_k)}(z_1,\dots,z_k) = \sum_{n_1>n_2>\cdots>n_k\geq 1} \frac{z_1^{n_1}\cdots z_k^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}},$$

and they satisfy not only shuffle relations, but also *stuffle relations*. When one specializes the stuffle relations in one variable at $z = 1$ and the stuffle relations in several variables at $z_1 = \cdots = z_k = 1$, one gets linear or quadratic dependence relations between the Multiple Zeta Values

$$\zeta(s_1,\dots,s_k) = \sum_{n_1>n_2>\cdots>n_k\geq 1} \frac{1}{n_1^{s_1}\cdots n_k^{s_k}}$$

which are defined for $k, s_1, \dots, s_k$ positive integers with $s_1 \geq 2$. *The Main Diophantine Conjecture* states that one obtains in this way all algebraic relations between these MZV.

0. Introduction

A long term project is to determine all algebraic relations among the values

$$\pi, \zeta(3), \zeta(5), \dots, \zeta(2n+1), \dots$$

of the Riemann zeta function

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}.$$

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So far, one only knows that the first number in this list, π, is transcendental, that the second one, $\zeta(3)$, is irrational, and that the other ones span a $\mathbb{Q}$-vector space of infinite dimension [Ri1], [BR]. See also [Ri2], [Zu1] and [Zu2].

The expected answer is disappointingly simple: it is widely believed that there are no relations, which means that these numbers should be algebraically independent:

(?) *For any* $n \geq 0$ *and any nonzero polynomial* $P \in \mathbb{Z}[X_0, \ldots, X_n]$,

$$P(\pi, \zeta(3), \zeta(5), \ldots, \zeta(2n+1)) \neq 0.$$

If true, this property would mean that there is no interesting algebraic structure.

The situation changes drastically if we enlarge our set so as to include the so-called Multiple Zeta Values (MZV, also called Euler-Zagier numbers or Polyzêta– see [Eu], [Z] and [C]):

$$\zeta(s_1, \ldots, s_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}},$$

which are defined for $k, s_1, \ldots, s_k$ positive integers with $s_1 \geq 2$. It may be hoped that the initial goal would be reached if one could determine all algebraic relations between the MZV. Now there are plenty of relations between them, providing a rich algebraic structure. One type of such relations arises when one multiplies two such series: it is easy to see that one gets a linear combination of MZV. There is another type of algebraic relations between MZV, coming from their expressions as integrals. Again the product of two such integrals is a linear combination of MZV. Following [B^3], we will use the name *stuffle* for the relations arising from the series, and *shuffle* for those arising from the integrals.

The *Main Diophantine Conjecture* (Conjecture 5.3 below) states that these relations are sufficient to describe all algebraic relations between MZV. One should be careful when stating such a conjecture: it is necessary to include some relations which are deduced from the stuffle and shuffle relations applied to divergent series (i.e. with $s_1 = 1$).

There are several ways of dealing with the divergent case. Here, we use the *multiple polylogarithms*

$$\mathrm{Li}_{(s_1, \ldots, s_k)}(z) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{z^{n_1}}{n_1^{s_1} \cdots n_k^{s_k}}$$

which are defined for $|z| < 1$ when $s_1, \ldots, s_k$ are all ≥ 1, and which are also defined for $|z| = 1$ if $s_1 \geq 2$.

These multiple polylogarithms can be expressed as iterated Chen integrals, and from this representation one deduces shuffle relations. There is no stuffle relations for multiple polylogarithms in a single variable, but one recovers them by introducing the multivariables functions

$$\mathrm{Li}_{(s_1,\dots,s_k)}(z_1,\dots,z_k)=\sum_{n_1>n_2>\cdots>n_k\geq 1}\frac{z_1^{n_1}\cdots z_k^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}},\quad (s_i\geq 1, 1\leq i\leq k)$$

which are defined not only for $|z_1|<1$ and $|z_i|\leq 1$ $(2\leq i\leq k)$, but also for $|z_i|\leq 1 (1\leq i\leq k)$ if $s_1\geq 2$.

1. Multiple Polylogarithms in One Variable and Multiple Zeta Values

Let $k, s_1,\dots,s_k$ be positive integers. Write $\underline{s}$ in place of $(s_1,\dots,s_k)$. One defines a complex function of one variable by

$$\mathrm{Li}_{\underline{s}}(z)=\sum_{n_1>n_2>\cdots>n_k\geq 1}\frac{z^{n_1}}{n_1^{s_1}\cdots n_k^{s_k}}.$$

This function is analytic in the open unit disk, and, in the case $s_1\geq 2$, it is also continuous on the closed unit disk. In the latter case we have

$$\zeta(\underline{s})=\mathrm{Li}_{\underline{s}}(1).$$

One can also define in an equivalent way these functions by induction on the number $p=s_1+\cdots+s_k$ (the *weight* of $\underline{s}$) as follows. Plainly we have

$$z\frac{d}{dz}\mathrm{Li}_{(s_1,\dots,s_k)}(z)=\mathrm{Li}_{(s_1-1,s_2,\dots,s_k)}(z)\quad\text{if}\quad s_1\geq 2 \tag{1.1}$$

and

$$(1-z)\frac{d}{dz}\mathrm{Li}_{(1,s_2,\dots,s_k)}(z)=\mathrm{Li}_{(s_2,\dots,s_k)}(z)\quad\text{if}\quad k\geq 2. \tag{1.2}$$

Together with the initial conditions

$$\mathrm{Li}_{\underline{s}}(0)=0, \tag{1.3}$$

the differential equations (1.1) and (1.2) determine all the $\mathrm{Li}_{\underline{s}}$.

Therefore, as observed by M. Kontsevich (cf. [Z]; see also [K] Chap. XIX, § 11 for an early reference to H. Poincaré, 1884), an equivalent definition for $\mathrm{Li}_{\underline{s}}$ is given by integral formulae as follows. Starting(*) with $k=s=1$, we write

$$\mathrm{Li}_1(z)=-\log(1-z)=\int_0^z\frac{dt}{1-t},$$

(*)This induction could as well be started from $k=0$, provided that we set $\mathrm{Li}_{\emptyset}(z)=1$.

where the complex integral is over any path from 0 to z inside the unit circle. From the differential equations (1.1) one deduces, by induction, for $s \geq 2$,

$$\mathrm{Li}_s(z) = \int_0^z \mathrm{Li}_{s-1}(t)\frac{dt}{t} = \int_0^z \frac{dt_1}{t_1}\int_0^{t_1}\frac{dt_2}{t_2}\cdots\int_0^{t_{s-2}}\frac{dt_{s-1}}{t_{s-1}}\int_0^{t_{s-1}}\frac{dt_s}{1-t_s}.$$

In the last formula, the complex integral over t_1, which is written on the left (and which is the last one to be computed), is over any path inside the unit circle from 0 to z, the second one over t_2 is from 0 to $t_1, \ldots,$ and the last one over t_s on the right, which is the first one to be computed, is from 0 to t_{s-1}.

Chen iterated integrals (see [K] Chap. XIX, § 11) provide a compact form for such expressions as follows. For $\varphi_1, \ldots, \varphi_p$ differential forms and x, y complex numbers, define inductively

$$\int_x^y \varphi_1\cdots\varphi_p = \int_x^y \varphi_1(t)\int_x^t \varphi_2\cdots\varphi_p.$$

For $\underline{s} = (s_1, \cdots s_k)$, set

$$\omega_{\underline{s}} = \omega_0^{s_1-1}\omega_1\cdots\omega_0^{s_k-1}\omega_1,$$

where

$$\omega_0(t) = \frac{dt}{t} \quad \text{and} \quad \omega_1(t) = \frac{dt}{1-t}.$$

Then the differential equations (1.1) and (1.2) with initial conditions (1.3) can be written

$$\mathrm{Li}_{\underline{s}}(z) = \int_0^z \omega_{\underline{s}}. \tag{1.4}$$

Example. Given a string $a_1, \ldots, a_k$ of integers, the notation $\{a_1, \ldots, a_k\}_n$ stands for the kn-tuple

$$(a_1, \ldots, a_k, \ldots, a_1, \ldots, a_k),$$

where the string $a_1, \ldots, a_k$ is repeated n times.

For any $n \geq 1$ and $|z| < 1$ we have

$$\mathrm{Li}_{\{1\}_n}(z) = \frac{1}{n!}(\log(1/(1-z)))^n, \tag{1.1}$$

which can be written in terms of generating series as

$$\sum_{n=0}^{\infty} \mathrm{Li}_{\{1\}_n}(z)x^n = (1-z)^{-x}.$$

The constant term $\mathrm{Li}_{\{1\}_0}(z)$ is 1.

2. Shuffle Product and the First Standard Relations

Denote by $X = \{x_0, x_1\}$ the alphabet with two letters and by X^* the set of words on X. A word is nothing else than a non-commutative monomial in the two letters x_0 and x_1. The linear combinations of such words with rational coefficients

$$\sum_u c_u u,$$

where $\{c_u; u \in X^*\}$ is a set of rational numbers with finite support, is the non-commutative ring $\mathfrak{H} = \mathbb{Q}\langle x_0, x_1\rangle$. The product is concatenation, the unit is the empty word e. We are interested with the set X^*x_1 of words which end with x_1. The linear combinations of such words is a left ideal of $\mathfrak{H}$ which we denote by $\mathfrak{H}x_1$. Also we denote by $\mathfrak{H}^1$ the subalgebra $\mathbb{Q}e + \mathfrak{H}x_1$ of $\mathfrak{H}$.

For s a positive integer, set $y_s = x_0^{s-1}x_1$. Next, for a tuple $\underline{s} = (s_1, \dots, s_k)$ of positive integers, define $y_{\underline{s}} = y_{s_1} \cdots y_{s_k}$. Hence the set X^*x_1 is also the set of words $y_{\underline{s}}$, where $k, s_1, \dots, s_k$ run over the set of all positive integers. We define $\widehat{\mathrm{Li}}_u(z)$ for $u \in X^*x_1$ by $\widehat{\mathrm{Li}}_u(z) = \mathrm{Li}_{\underline{s}}(z)$ when $u = y_{\underline{s}}$. By linearity we extend the definition of $\widehat{\mathrm{Li}}_u(z)$ to $\mathfrak{H}^1$:

$$\widehat{\mathrm{Li}}_u(z) = \sum_u c_u \widehat{\mathrm{Li}}_u(z) \quad \text{for} \quad v = \sum_u c_u u$$

where u ranges over a finite subset of $\{e\} \cup X^*x_1$ and $c_u \in \mathbb{Q}$, while $\widehat{\mathrm{Li}}_e(z) = 1$. The set of *convergent words* is the set, denoted by $\{e\} \cup x_0X^*x_1$, of words which start with x_0 and end with x_1 together with the empty word e. The $\mathbb{Q}$-vector subspace they span in $\mathfrak{H}$ is the subalgebra $\mathfrak{H}^0 = \mathbb{Q}e + x_0\mathfrak{H}x_1$ of $\mathfrak{H}^1$, and for v in $\mathfrak{H}^0$ we set

$$\hat{\zeta}(v) = \widehat{\mathrm{Li}}_v(1)$$

so that $\hat{\zeta} : \mathfrak{H}^0 \to \mathbb{R}$ is a $\mathbb{Q}$-linear map and

$$\hat{\zeta}(y_{\underline{s}}) = \zeta(\underline{s})$$

for $y_{\underline{s}}$ in $x_0X^*x_1$.

Definition. The *shuffle product* of two words in X^* is the element in $\mathfrak{H}$ which is defined inductively as follows:

$$e ш u = u ш e = u$$

for any u in X^*, and

$$(x_i u) ш (x_j v) = x_i(u ш (x_j v)) + x_j((x_i u) ш v)$$

for u, v in X^* and i, j equal to 0 or 1.

This product is extended by distributivity with respect to the addition to $\mathfrak{H}$ and defines a commutative and associative law. Moreover $\mathfrak{H}^0$ and $\mathfrak{H}^1$ are stable under ш. We denote by $\mathfrak{H}^0_ш \subset \mathfrak{H}^1_ш \subset \mathfrak{H}_ш$ the algebras where the underlying sets are $\mathfrak{H}^0 \subset \mathfrak{H}^1 \subset \mathfrak{H}$ respectively and the product is ш. Radford's Theorem gives the structure of these algebras: they are (commutative) polynomials algebras on the set of Lyndon words (see for instance [R]).

Computing the product $\widehat{\mathrm{Li}}_u(z)\widehat{\mathrm{Li}}_{u'}(z)$ of the two associated Chen iterated integrals yields (see [MPH], Th. 2):

Proposition 2.1. *For u and u' in $\mathfrak{H}^1_ш$,*

$$\widehat{\mathrm{Li}}_u(z)\widehat{\mathrm{Li}}_{u'}(z) = \widehat{\mathrm{Li}}_{uшu'}(z).$$

For instance from

$$x_1ш(x_0x_1) = x_1x_0x_1 + 2x_0x_1^2$$

we deduce

$$\mathrm{Li}_1(z)\mathrm{Li}_2(z) = \mathrm{Li}_{(1,2)}(z) + 2\mathrm{Li}_{(2,1)}(z). \tag{2.2}$$

Setting $z = 1$, we deduce from Proposition 2.1:

$$\hat{\zeta}(u)\hat{\zeta}(u') = \hat{\zeta}(uшu') \tag{2.3}$$

for u and u' in $\mathfrak{H}^0_ш$.

These are the *first standard relations* between multiple zeta values.

3. Shuffle Product for Multiple Polylogarithms in Several Variables

The functions of k complex variables(*)

$$\mathrm{Li}_{\underline{s}}(z_1, \dots, z_k) = \sum_{n_1>n_2>\cdots>n_k\geq 1} \frac{z_1^{n_1}\cdots z_k^{n_k}}{n_1^{s_1}\dots n_k^{s_k}}$$

have been considered as early as 1904 by N. Nielsen, and rediscovered later by A.B. Goncharov [G1, G2]. Recently, J. Écalle [É] used them for z_i roots of unity

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(*)Our notation for

$$\mathrm{Li}_{(s_1,\dots,s_k)}(z_1, \dots, z_k)$$

is the same as in [H], [W] or [C], but for Goncharov's [G2] it corresponds to

$$\mathrm{Li}_{(s_1,\dots,s_k)}(z_k, \dots, z_1).$$

(in case $s_1 \geq 2$): these are the *decorated multiple polylogarithms.* Of course one recovers the one variable functions $\mathrm{Li}_{\underline{s}}(z)$ by specializing $z_2 = \cdots = z_k = 1$. For simplicity we write $\mathrm{Li}_{\underline{s}}(\underline{z})$, where $\underline{z}$ stands for $(z_1, \dots, z_k)$. There is an integral formula which extends (1.4). Define

$$\omega_z(t) = \begin{cases} \frac{z\,dt}{1-zt} & \text{if } z \neq 0, \\ \frac{dt}{t} & \text{if } z = 0. \end{cases}$$

From the differential equations

$$z_1 \frac{\partial}{\partial z_1} \mathrm{Li}_{\underline{s}}(\underline{z}) = \mathrm{Li}_{(s_1-1, s_2, \dots, s_k)}(\underline{z}) \quad \text{if} \quad s_1 \geq 2$$

and

$$(1 - z_1) \frac{\partial}{\partial z_1} \mathrm{Li}_{(1, s_2, \dots, s_k)}(\underline{z}) = \mathrm{Li}_{(s_2, \dots, s_k)}(z_1 z_2, z_3, \dots, z_k)$$

generalizing (1.1) and (1.2), we deduce

$$\mathrm{Li}_{\underline{s}}(\underline{z}) = \int_0^1 \omega_0^{s_1-1} \omega_{z_1} \omega_0^{s_2-1} \omega_{z_1 z_2} \cdots \omega_0^{s_k-1} \omega_{z_1 \dots z_k}.$$

Because of the occurrence of the products $z_1 \dots z_j$ $(1 \leq j \leq k)$, it is convenient (see for instance [G1] and [B^{3}L]) to perform the change of variables

$$y_j = z_1^{-1} \cdots z_j^{-1} \;\; (1 \leq j \leq k) \;\; \text{and} \;\; z_j = \frac{y_{j-1}}{y_j} \;\; (1 \leq j \leq k)$$

with $y_0 = 1$, and to introduce the differential forms

$$\omega'_y(t) = -\omega_{y^{-1}}(t) = \frac{dt}{t - y},$$

so that $\omega'_0 = \omega_0$ and $\omega'_1 = -\omega_1$. Also define

$$\begin{aligned} \lambda \begin{pmatrix} s_1, \dots, s_k \\ y_1, \dots, y_k \end{pmatrix} &= \mathrm{Li}_{\underline{s}}(1/y_1, y_1/y_2, \dots, y_{k-1}/y_k) \\ &= \sum_{\nu_1 \geq 1} \cdots \sum_{\nu_k \geq 1} \prod_{j=1}^{k} y_j^{-\nu_j} \left(\sum_{i=j}^{k} \nu_i \right)^{-s_j} \\ &= (-1)^p \int_{\Delta_p} \omega_0^{s_1-1} \omega'_{y_1} \cdots \omega_0^{s_k-1} \omega'_{y_k}. \end{aligned}$$

With this notation some formulae are simpler. For instance the shuffle relation is easier to write with λ: the shuffle is defined on words on the alphabet $\{\omega'_y \,;\, y \in \mathbb{C}\}$, (including $y = 0$), inductively by

$$(\omega'_y u) ш (\omega'_{y'} v) = \omega'_y (u ш (\omega'_{y'} v)) + \omega'_{y'} ((\omega'_y u) ш v).$$

4. Stuffle Product and the Second Standard Relations

The functions $\mathrm{Li}_{\underline{s}}(\underline{z})$ satisfy not only shuffle relations, but also *stuffle relations* arising from the product of two series:

$$\mathrm{Li}_{\underline{s}}(\underline{z})\mathrm{Li}_{\underline{s}'}(\underline{z}') = \sum_{\underline{s}''} \mathrm{Li}_{\underline{s}''}(\underline{z}''), \tag{4.1}$$

where the notation is as follows: $\underline{s}''$ runs over the tuples $(s''_1, \dots, s''_{k''})$ obtained from $\underline{s} = (s_1, \dots, s_k)$ and $\underline{s}' = (\underline{s}'_1, \dots, s'_{k'})$ by inserting, in all possible ways, some 0 in the string $(s_1, \dots, s_k)$ as well as in the string $(s'_1, \dots, s'_{k'})$ (including in front and at the end), so that the new strings have the same length k'', with $\max\{k, k'\} \le k'' \le k + k'$, and by adding the two sequences term by term. For each such $\underline{s}''$, the component z''_i of $\underline{z}$ is z_j if the corresponding s''_i is just s_j (corresponding to a 0 in $\underline{s}'$), it is z'_ℓ if the corresponding s''_i is s'_ℓ (corresponding to a 0 in $\underline{s}$), and finally it is $z_j z'_\ell$ if the corresponding s''_i is $s_j + s'_\ell$. For instance

$$\begin{array}{cccccccc}
\underline{s} & s_1 & s_2 & 0 & s_3 & s_4 & \dots & 0 \\
\underline{s}' & 0 & s'_1 & s'_2 & 0 & s'_3 & \dots & s'_{k'} \\
\underline{s}'' & s_1 & s_2 + s'_1 & s'_2 & s_3 & s_4 + s'_3 & \dots & s'_{k'} \\
\underline{z}'' & z_1 & z_2 z'_1 & z'_2 & z_3 & z_4 z'_3 & \dots & z'_{k'}.
\end{array}$$

Of course the 0's are inserted so that no s''_i is zero.

Examples. For $k = k' = 1$ the shuffle relation (4.1) yields

$$\mathrm{Li}_s(z)\mathrm{Li}_{s'}(z') = \mathrm{Li}_{(s,s')}(z, z') + \mathrm{Li}_{s',s}(z', z) + \mathrm{Li}_{s+s'}(zz'), \tag{4.2}$$

while for $k = 1$ and $k' = 2$ we have

$$\begin{aligned}
\mathrm{Li}_s(z)\mathrm{Li}_{(s'_1,s'_2)}(z'_1, z'_2) &= \mathrm{Li}_{(s,s'_1,s'_2)}(z, z'_1, z'_2) + \mathrm{Li}_{(s'_1,s,s'_2)}(z'_1, z, z'_2) \\
&\quad + \mathrm{Li}_{(s'_1,s'_2,s)}(z'_1, z'_2, z) + \mathrm{Li}_{(s+s'_1,s'_2)}(zz'_1, z'_2) + \mathrm{Li}_{(s'_1,s+s'_2)}(z'_1, zz'_2).
\end{aligned} \tag{4.3}$$

The *stuffle product* $\star$ is defined on X^* inductively by

$$e \star u = u \star e = u$$

for $u \in X^*$,

$$x_0^n \star w = w \star x_0^n = \omega x_0^n$$

for any $n \ge 1$ and $w \in X^*$, and

$$(y_s u) \star (y_t u') = y_s(u \star (y_t u')) + y_t((y_s u) \star u') + y_{s+t}(u \star u')$$

for u and u' in X^*, $s \ge 1, t \ge 1$.

This product is extended by distributivity with respect to the addition to $\mathfrak{H}$ and defines a commutative and associative law. Moreover $\mathfrak{H}^0$ and $\mathfrak{H}^1$ are stable under $\star$. We denote by $\mathfrak{H}_\star^0 \subset \mathfrak{H}_\star^1 \subset \mathfrak{H}_\star$ the corresponding *harmonic* algebras. Their structure has been investigated by M. Hoffman [H]: again they are (commutative) polynomials algebras over Lyndon words.

Specializing (4.1) at $z_1 = \cdots = z_k = z'_1 = \cdots = z'_{k'} = 1$, we deduce

$$\hat{\zeta}(u)\hat{\zeta}(u') = \hat{\zeta}(u \star u') \tag{4.4}$$

for u and u' in $\mathfrak{H}_\star^0$.

These are the *second standard relations* between multiple zeta values. For instance (4.3) with $z = z'_1 = z'_2 = 1$ gives

$$\begin{aligned}\zeta(s)\zeta(s'_1, s'_2) &= \zeta(s, s'_1, s'_2) + \zeta(s'_1, s, s'_2) + \zeta(s'_1, s'_2, s)\\ &\quad + \zeta(s + s'_1, s'_2) + \zeta(s'_1, s + s'_2)\end{aligned}$$

for $s \geq 2$, $s'_1 \geq 2$ and $s'_2 \geq 1$.

5. The Third Standard Relations and the Main Diophantine Conjectures

We start with an example. Combining the stuffle relation (4.2) for $s = s' = 1$ with the stuffle relation (2.2) for $z' = z$, we deduce

$$\mathrm{Li}_{(1,2)}(z, 1) + 2\mathrm{Li}_{(2,1)}(z, 1) = \mathrm{Li}_{(1,2)}(z, z) + \mathrm{Li}_{(2,1)}(z, z) + \mathrm{Li}_3(z^2). \tag{5.1}$$

The two sides are analytic inside the unit circle, but not convergent at $z = 1$. We claim that

$$F(z) = \mathrm{Li}_{(1,2)}(z, 1) - \mathrm{Li}_{(1,2)}(z, z) = \sum_{n_1 > n_2 \geq 1} \frac{z^{n_1}(1 - z^{n_2})}{n_1 n_2^2}$$

tends to 0 as z tends to 1 inside the unit circle. Indeed for $|z| < 1$ we have

$$|1 - z^{n_2}| = |(1 - z)(1 + z + \cdots + z^{n_2 - 1})| < n_2 |1 - z|,$$

hence

$$\sum_{n_2=1}^{n_1-1} \frac{|1 - z^{n_2}|}{n_2^2} < |1 - z| \sum_{n_2=1}^{n_1-1} \frac{1}{n_2}.$$

From (1.5) with $n = 2$ we deduce

$$|F(z)| \leq |1 - z| \mathrm{Li}_{(1,1)}(|z|) = \frac{1}{2} |1 - z| (\log(1/(1 - |z|)))^2.$$

Therefore, taking the limit of the relation (5.1) as $z \to 1$ yields Euler's formula

$$\zeta(2, 1) = \zeta(3).$$

This argument works in a quite general setting and yields the relations

$$\hat{\zeta}(x_1 \star v - x_1 ш v) = 0 \tag{5.2}$$

for each $v \in \mathfrak{H}^0$.

These are the *third standard relations* between multiple zeta values.

The Main Diophantine Conjectures below arose after the works of several mathematicians, including D. Zagier, A.B. Goncharov, M. Kontsevich, M. Hoffman, M. Petitot and Hoang Ngoc Minh, K. Ihara and M. Kaneko (see [C]). They imply that the three standard relations (2.3), (4.4) and (5.2) generate the ideal of algebraic relations between all numbers $\zeta(\underline{s})$. Here are precise statements.

We introduce independent variables Z_u, where u ranges over the set $\{e\} \cup X^*_{x_1}$. For $v = \sum_u c_u u$ in $\mathfrak{H}^1$, we set

$$Z_v = \sum_u c_u Z_u.$$

In particular for u_1 and u_2 in $x_0 X^*_{x_1}$, $Z_{u_1 ш u_2}$ and $Z_{u_1 \star u_2}$ are linear forms in $Z_u, u \in x_0 X^* x_1$. Also, for $v \in x_0 X^* x_1$, $Z_{x_1 ш v - x_1 \star v}$ is a linear form in Z_u, $u \in x_0 X^* x_1$.

Denote by R the ring of polynomials with coefficients in $\mathbb{Q}$ in the variables Z_u where u ranges over the set $x_0 X^* x_1$, and by $\mathfrak{J}$ the ideal of R consisting of all polynomials which vanish under the specialization map

$$Z_u \mapsto \hat{\zeta}(u) \ (u \in x_0 X^* x_1).$$

Conjecture 5.3. The polynomials

$$Z_{u_1} Z_{u_2} - Z_{u_1 ш u_2}, \quad Z_{u_1} Z_{u_2} - Z_{u_1 \star u_2} \text{ and } Z_{x_1 \star v - x_1 ш v},$$

where u_1, u_2 and v range over the set of elements in $x_0 X^* x_1$, generate the ideal $\mathfrak{J}$.

This statement is slightly different from the conjecture in § 3 of [IK], where Ihara and Kaneko suggest that all linear relations between MZV's are supplied by the *regularized double shuffle relations*

$$\hat{\zeta}(\mathrm{reg}(u \star v - u ш v)) = 0,$$

where u ranges over $\mathfrak{H}^1$ and v over $\mathfrak{H}^0$. Here, reg is the $\mathbb{Q}$-linear map $\mathfrak{H} \to \mathfrak{H}^0$ which maps w to the constant term of the expression of w as a (commutative) polynomial in x_0 and x_1 with coefficients in $\mathfrak{H}^0_ш$. It is proved in [IK] that the linear polynomials $Z_{\mathrm{reg}(u \star v - u ш v)}$ associated to the regularized double shuffle relations belong to the ideal $\mathfrak{J}$. On the other hand, at least for the small weights, one can check that the regularized double shuffle relations follow from the three standard relations. Hence our Conjecture 5.3 seems stronger than the conjecture of [IK], but we expect they are in fact equivalent.

Denote by $\mathfrak{Z}_p$ the $\mathbb{Q}$-vector subspace of $\mathbb{R}$ spanned by the real numbers $\zeta(\underline{s})$ with $\underline{s}$ of weight p, with $\mathfrak{Z}_0 = \mathbb{Q}$ and $\mathfrak{Z}_1 = \{0\}$. Using any of the first two standard relations (2.3) or (4.4), one deduces $\mathfrak{Z}_p \cdot \mathfrak{Z}_{p'} \subset \mathfrak{Z}_{p+p'}$. This means that the $\mathbb{Q}$-vector subspace $\mathfrak{Z}$ of $\mathbb{R}$ spanned by all $\mathfrak{Z}_p$, $p \geq 0$, is a subalgebra of $\mathbb{R}$ over $\mathbb{Q}$ which is graded by the weight. From Conjecture 5.3 one deduces the following conjecture of Goncharov [G1]:

Conjecture 5.4. *As a $\mathbb{Q}$-algebra, $\mathfrak{Z}$ is the direct sum of $\mathfrak{Z}_p$ for $p \geq 0$.*

Conjecture 5.4 reduces the problem of determining all algebraic relations between MZV to the problem of determining all *linear* such relations. The dimension d_p of $\mathfrak{Z}_p$ satisfies $d_0 = 1, d_1 = 0, d_2 = d_3 = 1$. The expected value for d_p is given by a conjecture of Zagier [Z]:

Conjecture 5.5. *For $p \geq 3$ we have*

$$d_p = d_{p-2} + d_{p-3}.$$

An interesting question is whether Conjecture 5.3 implies Conjecture 5.5. For this question as well as other related problems, see [É] and [C].

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A (Conjectural) 1/3-phenomenon for the Number of Rhombus Tilings of a Hexagon which Contain a Fixed Rhombus

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We state, discuss, provide evidence for, and prove in special cases the conjecture that the probability that a random tiling by rhombi of a hexagon with side lengths $2n+a$, $2n+b$, $2n+c$, $2n+a$, $2n+b$, $2n+c$ contains the (horizontal) rhombus with coordinates $(2n+x, 2n+y)$ is equal to $\frac{1}{3}+g_{a,b,c,x,y}(n)\binom{2n}{n}^3/\binom{6n}{3n}$, where $g_{a,b,c,x,y}(n)$ is a rational function in n. Several specific instances of this "1/3-phenomenon" are made explicit.

1. Introduction and Statement of the Conjecture

Let a, b and c be positive integers, and consider a hexagon with side lengths a, b, c, a, b, c whose angles are 120° (see Figure 1.a for an example). The subject of our interest is the enumeration of tilings of this hexagon by rhombi (cf. Figure 1.b; here, and in the sequel, by a rhombus we always mean a rhombus with side lengths 1 and angles of 60° and 120°).

As is well-known, the total number of rhombus tilings of a hexagon with side lengths a, b, c, a, b, c equals

$$\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}.$$

(This follows from MacMahon's enumeration [14, Sec. 429, $q\to 1$; proof in Sec. 494] of all plane partitions contained in an $a\times b\times c$ box, as these are in

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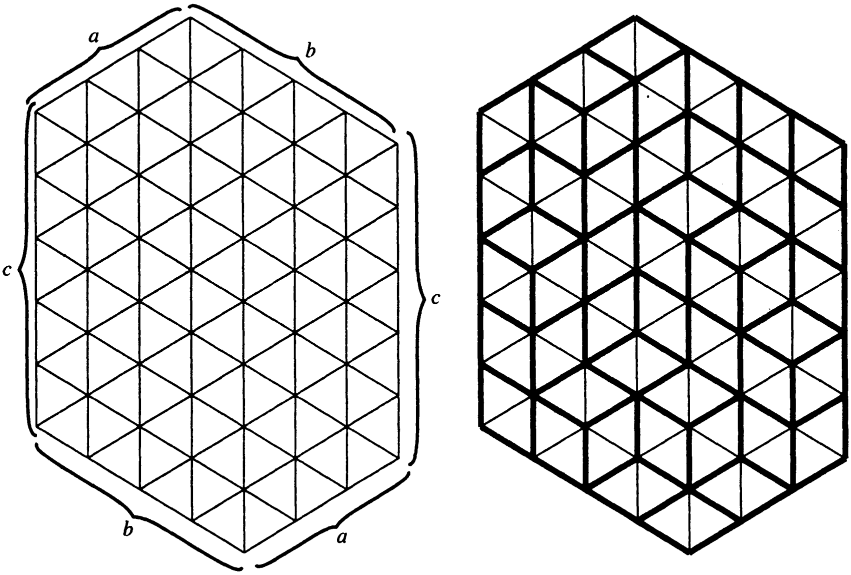

a. A hexagon with sides a, b, c, a, b, c where $a = 3, b = 4, c = 5$

b. A rhombus tiling of a hexagon with sides a, b, c, a, b, c

Figure 1

bijection with rhombus tilings of a hexagon with side lengths a, b, c, a, b, c, as explained e.g. in [4].)

The problem that we are going to address in this paper is the problemof enumerating rhombus tilings of a hexagon which contain a given fixed rhombus. Since the total number of rhombus tilings of a given hexagon is known, thanks to MacMahon's formula, we may ask equivalently the question of what the probability is that a rhombus tiling of a hexagon that is chosen uniformly at random (to be precise, it is the *tiling* which is chosen at random, while the *hexagon* is given) contains a given fixed rhombus. (For example, we may ask what the probability is that a randomly chosen rhombus tiling of the hexagon with side lengths $3, 5, 4, 3, 5, 4$, shown in Figure 4, contains the shaded rhombus. At this point the thick lines are without relevance.)

If this question is asked for an "infinite" hexagon, i.e., if we imagine the 2-dimensional plane being covered by a triangular grid (each triangle being an equilateral triangle; see Figure 2; at this point shades in the figure should be ignored), and ask the question of what the probability is that a particular rhombus formed out of two adjacent triangles (for example the shaded rhombus in Figure 2) is contained in a randomly chosen rhombus tiling of the plane (that is compatible with the triangular grid, of course), then there is a simple argument which shows that this probability is $1/3$: Let us concentrate on one of the two adjacent

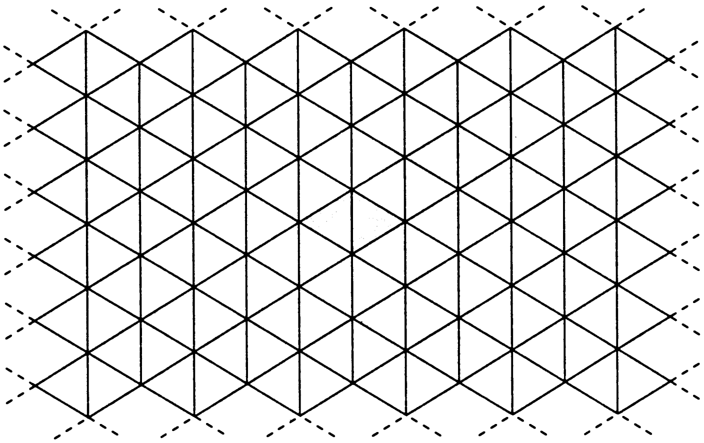

Figure 2

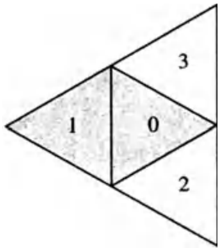

Figure 3

triangles out of which our fixed rhombus is formed. (In Figure 3 we have enlarged the chosen rhombus. It is composed out of the triangles labelled 0 and 1. We are going to concentrate on the triangle labelled 0.) This triangle is adjacent to exactly three other triangles. (In Figure 3 these are the triangles labelled 1, 2 and 3.) In a rhombus tiling this triangle must be combined with exactly one of these to form a rhombus in the tiling. Hence, the probability that a *random* tiling will combine the triangle with the particular one to obtain the fixed rhombus is $1/3$.

For a (finite) hexagon however, we must expect a very different behaviour, resulting from the boundary of the hexagon. The probability that a particular rhombus is contained in a random tiling will heavily depend on where the rhombus is located in the hexagon. (This is for example reflected in the asymptotic result of Cohn, Larsen and Propp [3, Theorem 1].) In particular, we must expect that the probability will usually be different from $1/3$.

Rather surprisingly, John[1] [11, bottom of p. 198] and Propp [17, 18, Problem 1] made the empirical observation that in a hexagon with side lengths $2n-1, 2n-1, 2n, 2n-1, 2n-1, 2n$ the probability that the central rhombus is contained in a random tiling is *exactly* $1/3$, the same being apparently true in a hexagon with side lengths $2n, 2n, 2n-1, 2n, 2n, 2n-1$. These facts were proved by Ciucu and the author [2, Cor. 3] and, independently, by Helfgott and Gessel [9, Theorem 17]. In fact, more generally, in both papers the probability that in a hexagon with side lengths N, N, M, N, N, M, where $N \not\equiv M \bmod 2$, the central rhombus is contained in a random tiling is expressed in terms of a single sum, from which the $1/3$-result follows on simplification of the sum. These results were generalized in two directions. On the one hand, Fulmek and the author [6] found a single sum expression for this probability for *any* rhombus on the (horizontal) symmetry axis of the hexagon. On the other hand, Fischer [5] gave a single sum expression for the probability that the central rhombus is contained in a random tiling of a hexagon with *arbitrary* side lengths (i.e., with side lengths a, b, c, a, b, c). Some further single sum expressions for probabilities of "near-central" rhombi to be contained in a random tiling of a hexagon with sides N, N, M, N, N, M have been derived in [5, Theorem 2] and [7]. Finally, in complete generality, Fischer [5, Lemma 2] and Johansson [10, (4.37)] found *triple sum* expressions for the probability that a fixed (not necessarily central or near-central) rhombus is contained in a random tiling of a hexagon with side lengths a, b, c, a, b, c. (These two triple sum expressions are completely different from each other.)

The purpose of this paper is to report a curious manifestation of the fact that "in the limit" the probability that a particular rhombus is contained in a random tiling is $1/3$. Roughly speaking, it seems that the probability equals

$$1/3 \text{ plus a "nice" expression.}$$

To make this precise, we need to introduce a convention of how to describe the position of a rhombus in a given hexagon. First of all, without loss of generality, we may restrict our considerations to the case where the fixed rhombus is a horizontal rhombus (by which we mean a rhombus such as the shaded ones in Figures 2–4), which we shall do for the rest of the paper. (The other two types of rhombi are then covered via a rotation by 120°, respectively by 240°.) In order to describe the position of a rhombus in the hexagon, we introduce, following [5], the following oblique angled coordinate system: Its origin is located in one of the two vertices

---

[1] In fact, in [11] the problem of finding the probability that, given a hexagonal graph, a chosen fixed edge is contained in a randomly chosen perfect matching of the graph is dealt with. The motivation to consider this problem is that such hexagonal graphs serve as models for benzenoid hydrocarbon molecules. The above probability is called *Pauling's bond order.* It measures how stable a carbon-carbon bond (corresponding to the fixed edge) in a benzenoid hydrocarbon molecule is.

It is well-known that this problem is equivalent to our tiling problem. The link is a bijection between rhombus tilings of a fixed subregion of the infinite triangular grid (such as our hexagons) and perfect matchings of the hexagonal graph which is, roughly speaking, the dual graph of the subregion (see e.g. [12]; "roughly speaking" refers to the little detail that the vertex corresponding to the outer face is ignored in the dual graph construction).

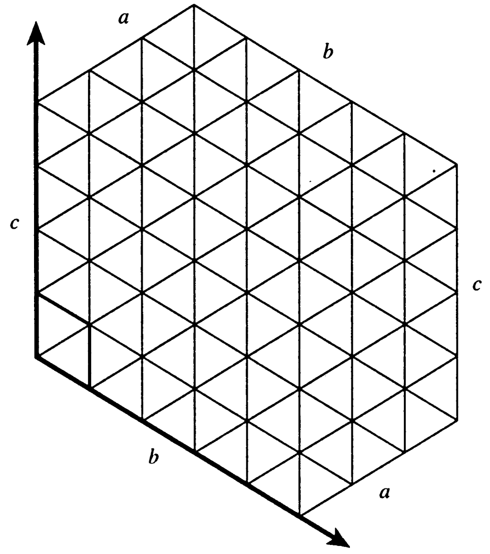

Figure 4 The oblique angled coordinate system

where the sides of lengths b and c meet, and the axes are induced by those two sides (see Figure 4). The units are chosen such that the grid points of the triangular grid are exactly the integer points in this coordinate system. (That is to say, the two triangles in Figure 4 with vertices in the origin form the unit "square.") Thus, in this coordinate system, the bottom-most point of the shaded hexagon in Figure 4 has coordinates $(5, 4)$.

With this convention, we have the following conjecture. It extends an (ex) conjecture by Propp [17, 18, Problem 4].

Conjecture. Let a, b, c, x and y be arbitrary integers. Then the probability that a randomly chosen rhombus tiling of a hexagon with side lengths $2n + a$, $2n + b$, $2n + c$, $2n + a$, $2n + b$, $2n + c$ contains the (horizontal) rhombus with bottom-most vertex $(2n + x, 2n + y)$ (in the oblique angled coordinate system) is equal to

$$\frac{1}{3} + f_{a,b,c,x,y}(n)\binom{2n}{n}^3 \bigg/ \binom{6n+2}{3n+1} \quad \text{for } n > n_0, \tag{1.1}$$

for a suitable n_0 which depends on a, b, c, x and y, where $f_{a,b,c,x,y}(n)$ is a rational function in n.[2]

As is shown in Section 2, for any *specific* a, b, c, x, y the corresponding formula for $f_{a,b,c,x,y}(n)$ can be worked out completely automatically by the use of a computer (given that the Conjecture is true, of course). We have in fact produced a huge list of such formulas, of which we list a few selected instances below. As we explain in Section 3, any of these is (at least) a "near-theorem," in the sense that it could be proved automatically by the available multisum algorithms, provided there is enough computer memory available (and, thus, will at least be a theorem in the near future). Also in Section 3, we elaborate more precisely on which of these are just conjectural, and which of them are already theorems[3]. However, we do not know how to prove the Conjecture *in general*, that is, for *generic* values of a, b, c, x, and y (cf. Section 3 for a possible approach).

---

[2] This statement is clearly equivalent to the statement in the abstract. The form (1.1) of the expression is more convenient in the subsequent listing of special cases.

Here is the announced excerpt from our list of special instances of the Conjecture:

$$f_{-1,-1,0,-1,-1}(n) = f_{2,2,1,2,1}(n) = 0 \qquad \text{for } n \geq 1, \tag{1.2}$$

$$\begin{aligned} f_{2,1,1,2,1}(n) &= f_{2,1,1,1,1}(n) = f_{1,2,1,2,1}(n) = f_{1,2,1,1,0}(n) \\ &= f_{-1,0,0,0,-1}(n) = f_{-1,0,0,-1,-1}(n) = f_{0,-1,0,0,0}(n) \\ &= f_{0,-1,0,-1,-1}(n) = 0 \qquad \text{for } n \geq 1, \end{aligned} \tag{1.3}$$

$$\begin{aligned} f_{1,1,1,1,1}(n) &= f_{1,1,0,1,-1}(n) = f_{1,1,0,1,1}(n) \\ &= f_{1,1,1,0,-1}(n) = f_{1,1,1,0,1}(n) = f_{1,1,1,1,0}(n) \\ &= f_{0,2,0,1,0}(n) = f_{2,0,0,0,1}(n) = f_{2,0,0,1,0}(n) \\ &= f_{2,0,0,1,1}(n) = f_{2,0,0,2,0}(n) = \frac{1}{3} \qquad \text{for } n \geq 1, \end{aligned} \tag{1.4}$$

$$f_{1,1,0,0,-1}(n) = f_{1,1,0,0,0}(n) = f_{2,0,1,1,0}(n) = \frac{1}{3} \qquad \text{for } n \geq 1, \tag{1.5*}$$

$$f_{0,2,1,1,0}(n) = -\frac{2}{3} \qquad \text{for } n \geq 1, \tag{1.6}$$

$$f_{1,1,1,0,0}(n) = -\frac{2}{3} \qquad \text{for } n \geq 1, \tag{1.7*}$$

$$f_{1,1,0,1,0}(n) = \frac{4}{3} \qquad \text{for } n \geq 1, \tag{1.8}$$

$$f_{4,3,1,3,2}(n) = f_{4,3,1,4,2}(n) = \frac{4}{3} \qquad \text{for } n \geq 1, \tag{1.9*}$$

$$f_{0,0,0,0,0}(n) = -\frac{(6n+1)}{6(3n+1)} \qquad \text{for } n \geq 1, \tag{1.10}$$

---

[3] For the convenience of the reader, we have marked conjectures by an asterisk in the equation number.

$$f_{0,0,1,0,0}(n) = -\frac{2(6n+1)}{3(3n+1)} \qquad \text{for } n \geq 1, \tag{1.11}$$

$$f_{3,3,0,3,1}(n) = \frac{2(2n+1)(3n+2)(4n+5)}{3(n+1)^2(6n+5)} \qquad \text{for } n \geq 1, \tag{1.12}$$

$$f_{2,1,0,3,-1}(n) = \frac{4n^3+18n^2+12n+1}{6(n+1)^2(2n-1)} \qquad \text{for } n \geq 2, \tag{1.13*}$$

$$f_{5,1,0,3,2}(n) = \frac{(3n+2)(16n^3+54n^2+57n+20)}{3(n+1)^2(n+2)(6n+5)} \qquad \text{for } n \geq 1, \tag{1.14*}$$

$$f_{-1,5,0,2,-1}(n) = \frac{(3n+2)(2n^2+4n+1)}{3(n+1)^2(n+2)} \qquad \text{for } n \geq 1, \tag{1.15*}$$

$$\begin{aligned} &f_{10,3,0,1,4}(n) \\ &\quad = \frac{(2n+1)(2n+3)(3n+2)(3n+4)(3n+5)}{6(n+1)^2(n+2)^2(n+3)^2(n+4)^2(n+5)(2n-3)(2n-1)(6n+5)(6n+7)(6n+11)} \\ &\quad\quad \times(176n^9+3080n^8+21692n^7+74546n^6+102578n^5-73279n^4-362598n^3 \\ &\quad\quad -283977n^2+24762n+55440) \qquad \text{for } n \geq 2. \end{aligned} \tag{1.16*}$$

2. How are these Conjectures and Results Discovered?

Point of departure for all these discoveries is an observation by Propp [17, 18, Problem 4]: He conjectured that the probability that a randomly chosen rhombus tiling of a hexagon with *all* side lengths equal to N contains the "near-central" rhombus (this is the rhombus with bottom-most vertex (N, N) in the oblique angled coordinate system) is equal to $1/3$ plus a "nice" formula in N.[4] Should this observation be true, then the *Mathematica* program `Rate`[5] ("Rate!" is German for "Guess!"), respectively its *Maple* equivalent GUESS[6], will find the formula, given enough initial terms of the sequence.

Let us see how this works in the case that N is odd. For the generation of the probabilities, Propp used the programs `vaxmaple`[7] and `vaxmacs`[8], which are based on the evaluation of determinants of large (if though sparse) matrices.

---

[4] Commonly, by a "nice" formula one means an expression which is built by forming products and quotients of factorials. A strong indication that one encounters a sequence $(a_N)_{N\geq 0}$ for which a "nice" formula exists is that the prime factors in the prime factorization of a_N do not grow rapidly as N becomes larger. (In fact, they should grow linearly.)

[5] written by the author; available from `http://radon.mat.univie.ac.at/People/kratt`; see [13, Appendix A] for an explanation of how the program works.

[6] written by François Béraud and Bruno Gauthier; available from `http://www-igm.univ-mlv.fr/~gauthier`.

[7] written by Greg Kuperberg, Jim Propp and David Wilson; available at `http://math.wisc.edu/~propp/software.html`.

[8] written by David Wilson; also available at `http://math.wisc.edu/~propp/software.html`.

However, since then triple sum formulas have been found by Fischer [5, Lemma 2] and Johansson [10, (4.37)], which allow to generate these probabilities much more efficiently. We choose to use Fischer's formula. We state it below.

Theorem. *Let a, b and c be positive integers, and let (x, y) be an integer point such that $0 \le x \le b + a - 1$ and $1 \le y \le c + a - 1$. Then the probability that a randomly chosen rhombus tiling of a hexagon with side lengths a, b, c, a, b, c contains the (horizontal) rhombus with bottom-most vertex (x, y) (in the oblique angled coordinate system) is equal to*

$$\begin{gathered}\frac{c!}{(b+1)_c}\sum_{i=1}^{a}\sum_{j=1}^{a}\sum_{s=1}^{j}\\ (-1)^{i+s}\binom{j-1}{s-1}\binom{c+i+x-y-2}{x-1}\binom{b+s-x+y-1}{b+s-x-1}\\ \cdot\frac{(b+1)_{s-1}\,(c+1)_{i-1}\,(b+c+i)_{j-i}}{(j-i)!\,(i-1)!\,(b+c+1)_{s-1}}.\end{gathered}\tag{2.1}$$

We now program this formula in *Mathematica.*

```
Mathematica 2.2 for DOS 387

Copyright 1988-93 Wolfram Research, Inc.

In[1]:= F[a_,b_,c_,x_,y_]:=c!/Pochhammer[b+1,c]*
   Sum[Sum[Sum[(-1)^(i+s)*Binomial[j-1,s-1]*
     Binomial[c+i+x-y-2,x-1]*Binomial[b+s-x+y-1,b+s-x-1]*
     Pochhammer[b+1,s-1]*Pochhammer[c+1,i-1]*
     Pochhammer[b+c+i,j-i]/(j-i)!/(i-1)!/
     Pochhammer[b+c+1,s-1],
   {s,1,j}],{j,1,a}],{i,1,a}]
```

Now we generate the first eleven values of these probabilities for $N = 2n + 1$ and subtract $1/3$ from them.

```
In[2]:= Table[F[2n+1,2n+1,2n+1,2n+1,2n+1]-1/3,{n,1,11}]

          4    3    2000    245    296352    142296
Out[2]= {---, ---, ------, -----, --------, --------,
         105  143  138567  22287  33393355  19126225

      43188288    759169125     15365378600
>    ----------, ------------, -------------,
     6743906935  135054066707  3067656658059

      55469016746     805693639296
>    --------------, ---------------}
     12280863528759  195909013434965
```

Next we load `Rate`, and apply `Rate`'s function `Ratekurz` to the sequence of numbers.

```
In[3]:= <<rate.m

In[4]:= Apply[Ratekurz,%2]

                                     2
         4          (1 + 2 i1)  (2 + 3 i1) (4 + 3 i1)
Out[4]= {--- Product[---------------------------------,
        105                     2
                       (1 + i1)  (5 + 6 i1) (7 + 6 i1)
>          {i1, 1, -1 + i0}}]
```

The program outputs a formula which generates the terms of the sequence that was given as an input. The formula is written as a function in `i0`, i.e., we must replace `i0` by n. In more compact terms, the formula can be rewritten as

$$\frac{1}{3}\binom{2n}{n}^3 \Big/ \binom{6n+2}{3n+1}. \tag{2.2}$$

(It should be observed that this expression is exactly the one which features in (1.1).)

At this point, this formula is of course just a conjecture. It has however been proved in [7, Corollary 7, (1.9)].

Being adventurous, one tries the same thing for other choices of the parameters a, b, c, x and y. Very quickly one discovers, that a similar phenomenon seems to occur for any choice $2n+a$, $2n+b$, $2n+c$ for the side lengths and $(2n+x, 2n+y)$ for the coordinates of the bottom-most point of the fixed rhombus, where a, b, c, x, and y are fixed integers. Although the (conjectural) expressions that one finds need not be "nice" anymore in the strict sense above, it is at worst polynomial factors in n that appear in addition. Moreover, one also realizes soon that division of such an expression by the expression in (2.2) apparently always results in a rational function in n, i.e., the Conjecture in Section 1 is discovered.

Let us see just one such example. We choose a hexagon with side lengths $2n+2$, $2n+1$, $2n$, $2n+2$, $2n+1$, $2n$, and $(2n+3, 2n-1)$ for the coordinates of the bottom-most point of the fixed rhombus. Then we obtain the following numbers for $n = 1, 2, \dots, 15$. The reader should note that we immediately divide the expression (2.2).

```
In[5]:= Table[(F2[2n+2,2n+1,2n,2n+3,2n-1]-1/3)/
          (Binomial[2n,n]^3/Binomial[6n+2,3n+1]),{n,1,15}]
```

```
           35   43  307  593   337  1585  2339  1099
Out[5]= {-(--), --, ---, ----, ---, ----, ----, ----,
           12   54  480  1050  648  3234  4992  2430

     4483   5921   2545  9649   11987  4891   17731
>    -----, -----, ----, -----, -----, -----, -----}
     10200  13794  6048  23322  29400  12150  44544
```

By having a brief glance at this sequence, it seems that the first term is “alien,” so let us better drop it.

```
In[6]:= Drop[%,1]
         43  307  593   337  1585  2339  1099  4483   5921
Out[6]= {--, ---, ----, ---, ----, ----, ----, -----, -----,
         54  480  1050  648  3234  4992  2430  10200  13794

     2545  9649   11987  4891   17731
>    ----, -----, -----, -----, -----}
     6048  23322  29400  12150  44544
```

By the discussion above, this should be a sequence which is given by a rational function in n. Therefore is suffices to apply `Rate`’s `Rateint` (which does just rational interpolation, in contrast to `Ratekurz`, which tries several other things, and which is therefore slower).

```
In[7]:= Apply[Rateint,%]
                           2       3
         35 + 60 i0 + 30 i0  + 4 i0
Out[7]= {---------------------------}
                     2
           6 (2 + i0)  (1 + 2 i0)
```

Again, the program outputs the formula as a function in `i0`. Since initially we dropped the first term of the sequence, we must now replace `i0` by $n - 1$.

```
In[8]:= Factor[%/.i0->n-1]
                        2      3
         1 + 12 n + 18 n  + 4 n
Out[8]= {-----------------------}
                   2
          6 (1 + n)  (-1 + 2 n)
```

Hence, if the Conjecture in Section 1 is true, $f_{2,1,0,3,-1}(n)$ must be the expression given in the output `Out[8]`. (Thus, we have discovered Eq. (1.13).) Again, at this point, this is just a conjecture.

The Equations (1.2)–(1.16) in Section 1 are all found in the same way.

3. Discussion: How to Prove the Conjecture?

A possible approach to prove the Conjecture in Section 1 is to start with the expression (2.1) (or with the alternative expression [10, (4.37)]), replace a by $2n+a$, b by $2n+b$, c by $2n+c$, x by $2n+x$, y by $2n+y$, and by some manipulation (for example, by applying hypergeometric transformation and summation formulas) convert it into the form (2.2). Everybody who has some experience with manipulating binomial/hypergeometric sums will immediately realize that this is a formidable task. In particular, it seems a bit mysterious how one should be able to isolate "1/3" from the "rest." In any case, I do not know how to prove the Conjecture in this manner, nor in any other way.

On the other hand, as we explained in Section 2, for any *specific* values of a, b, c, x, and y, it is routine to find a conjectural expression for the rational function $f_{a,b,c,x,y}(n)$ (given that the Conjecture is true). In turn, once such an expression is available, it can (at least in principle) be verified completely automatically. For, what one has to prove is the equality of the expressions (2.1), with the above replacements, and (1.1), where $f_{a,b,c,x,y}(n)$ is the explicit rational function found by the computer. That is to say, one has to prove that a certain triple sum equals a closed form expression. Clearly, this can be done (again, at least in principle) by the available multisum algorithms[9], by using the algorithm to find a recurrence in n for the expression (2.1), and subsequently checking that the expression (1.1), with the computer guess for $f_{a,b,c,x,y}(n)$, satisfies the same recurrence. Unfortunately, in any case that I tried, the computer ran out of memory.

However, as we already mentioned in the Introduction, in some cases formulas in form of single hypergeometric sums are available. If one is in such a case then one would proceed as in the above paragraph, but one would replace the multisum algorithm by Zeilberger's algorithm[10] (see [15, 16, 23, 24]). The advantage is that, in contrast to the multisum algorithm, Zeilberger's algorithm is very efficient. At any rate, in any case that I looked at in connection with our problem, the Zeilberger algorithm was successful. That is to say, if I am allowed to somewhat overstate it, ***whenever one is in a case where a single sum formula is available, one has a theorem*** (i.e., Zeilberger's algorithm *will* prove that the empirical found rational function $f_{a,b,c,x,y}(n)$ does indeed satisfy the Conjecture for *all* values of n).

---

[9] The first (theoretical) algorithm for proving multisum identities automatically was given by Wilf and Zeilberger [22]. A considerable enhancement and speedup was accomplished by Wegschaider [21], who combined the ideas of Wilf and Zeilberger with ideas of Verbaeten [20]. Wegschaider's *Mathematica* implementation is available from `http://www.risc.uni-linz.ac.at/research/combinat/risc/software`.

[10] A *Maple* implementation written by Doron Zeilberger is available from `http://www.math.temple.edu/~zeilberg`; a *Mathematica* implementation written by Markus Schorn and Peter Paule is available from `http://www.risc.uni-linz.ac.at/research/combinat/risc/software`.

For the sake of completeness, we list the vectors (a, b, c, x, y) for which single sums are available for $f_{a,b,c,x,y}(n)$. Clearly, it suffices to restrict c to 0 and 1. (All other values can be attained by shifts of n.)

(A) By [5, Theorem 1, (1.2)]: $(2a'+1, 2b'+1, 0, a'+b'+1, a')$, for integers a' and b'.
(B) By [5, Theorem 1, (1.3)]: $(2a', 2b', 1, a'+b', a')$, for integers a' and b'.
(C) By [5, Theorem 2, (1.4)]: $(2a'+1, 2b'+1, 1, a'+b'+1, a'+1)$ and $(2a'+1, 2b'+1, 1, a'+b'+1, a')$, for integers a' and b'.
(D) By [5, Theorem 2, (1.5)]: $(2a', 2b', 0, a'+b', a')$ and $(2a', 2b', 0, a'+b', a'-1)$, for integers a' and b'.
(E) By [6, Theorem 1]: $(a', a', 0, 2x'+1, x')$, for integers a' and x'.
(F) By [6, Theorem 2]: $(a', a', 1, 2x', x'-1)$, for integers a' and x'.
(G) By [7, Theorem 3]: $(2a', 2a', 1, 2a', a'+1)$ and $(2a', 2a', 1, 2a', a'-1)$, for an integer a'.
(H) By [7, Theorem 4]: $(2a'+1, 2a'+1, 0, 2a'+1, a'+1)$ and $(2a'+1, 2a'+1, 0, 2a'+1, a'-1)$, for an integer a'.
(I) By [7, Theorem 5]: $(2a', 2a', 0, 2a', a'+1)$ and $(2a', 2a', 0, 2a', a'-2)$, for an integer a'.
(J) By [7, Theorem 6]: $(2a'+1, 2a'+1, 1, 2a'+1, a'+2)$ and $(2a'+1, 2a'+1, 1, 2a'+1, a'-1)$, for an integer a'.

Thus, choosing $a' = b' = 0$ in (A), we see for example that the expression for $f_{0,0,1,0,0}(n)$ given in (1.11), is in fact a theorem. For, by Theorem 1, (1.3) in [5] with $a = b = 2n$, $c = 2n+1$, the probability that a randomly chosen rhombus tiling of a hexagon with side lengths $2n, 2n, 2n+1, 2n, 2n, 2n+1$ contains the (horizontal) rhombus with bottom-most vertex $(2n, 2n)$ can be written in the form

$$\text{SUM}(n) := \sum_{k=0}^{n-1} \frac{2n\,(2n+1)!}{(2n+1)_{4n}} \binom{2n}{n} \binom{3n}{n} 2^{2n-2} \cdot (n+3/2)_k\,(2n+1)_k\,(n+k+2)_{n-k-1} \cdot (2n+k+2)_{n-k-1} \frac{(1/2)_{n-k-1}}{(n-k-1)!}. \tag{3.1}$$

Next we take it as an input for Zeilberger's algorithm (we are using Zeilberger's *Maple* implementation here):

```
   |\^/|      Maple V Release 4 (Uni Wien)
._|\|   |/|_.  Copyright (c) 1981-1996 by Waterloo Maple Inc.  All rights
 \  MAPLE  /  reserved.  Maple and Maple V are registered trademarks of
 <____ ____>  Waterloo Maple Inc.
       |      Type ?  for help.
> read ekhad:

> ezra(zeillim);
                       zeillim(SUMMAND,k,n,N,alpha,beta)
         Similar to zeil(SUMMAND,k,n,N) but outputs a recurrence for
                 the sum of SUMMAND from k=alpha to k=n-beta
       Outputs the recurrence operator, certificate and right hand side.
       For example, "zeillim(binomial(n,k),k,n,N,0,1);" gives output of
                               N-2, k/(k-n-1),1
            which means that SUM(n):=2^n-1 satisfies the recurrence
               (N-2)SUM(n)=1, as certified by R(n,k):=k/(k-n-1)

> zeillim(2*n*(2*n+1)!/rf(2*n+1,4*n)*binomial(2*n,n)*binomial(3*n,n)*
> 2^(2*n-2)*rf(n+3/2,k)*rf(2*n+1,k)*rf(n+k+2,n-k-1)*
> rf(2*n+k+2,n-k-1)*rf(1/2,n-k-1)/(n-k-1)!,k,n,N,0,1);

-1 + N, 1/6 (1 - 2 n + 2 k) (-288 n   - 432 n  k - 912 n   - 1440 n  k

            4  2        3  2         3           3        2  2        2
     - 216 n  k  - 414 n  k  - 1746 n  k - 1096 n  - 189 n  k  - 612 n

           2  3        2           3                           2
     + 36 n  k  - 907 n  k + 48 n k  - 152 n - 163 n k + 43 n k  - 12 + 4 k

           2       3    /
     + 32 k  + 16 k )  /  ((-n + k) n (6 n + 1) (6 n + 5) (2 n + k + 2)
                      /

                  2
    (1 + 3 n + 2 n )),

                  2               3   n      3       2
        GAMMA(3 n)  GAMMA(n + 1/2)  64  (36 n  + 60 n  + 29 n + 3)
    1/2 ----------------------------------------------------------
               2   3/2         3                      2
        (n + 1)  Pi    GAMMA(n)  (6 n + 5) (6 n + 1) n  GAMMA(6 n)
```

It tells us that the expression SUM(n) in (3.1) satisfies the recurrence

$$\mathrm{SUM}(n+1)-\mathrm{SUM}(n) = \frac{(3+29n+60n^2+36n^3)\,(3n-1)!^2\,(2n)!^3}{2n^2\,(n+1)^2\,(6n+1)\,(6n+5)\,(n-1)!^3\,(6n-1)!\,n!^3}. \tag{3.2}$$

(The first term in the output, `-1 + N`, encodes the form of the left-hand side of (3.2), the third term gives the right-hand side. The middle term is the so-called *certificate* which provides a proof of the recurrence.) So it just remains to check that the expression (1.1) with $f_{0,0,1,0,0}(n)$ as in (1.11) satisfies the same recurrence and agrees with (3.1) for $n = 1$, which is of course a routine task.

On the other hand, the expression for $f_{2,1,1,2,1}(n)$ given in (1.3) cannot be established in the same way by appealing to a special case of one of (A)–(H). Still, it is also a theorem, thanks to the following simple observation: suppose that we consider a hexagon with side lengths a, b, c, a, b, c, where $a = b$, and a rhombus on the horizontal symmetry axis of the hexagon. Let us imagine that this rhombus were the one in Figure 3 (consisting of the triangles labelled 0 and 1). Let us denote the probability that a randomly chosen tiling contains this rhombus by p. Since the rhombus is on the symmetry axis, the probability that a randomly chosen tiling contains the rhombus consisting of the triangles labelled 0 and 2 is equal to the probability that it contains the rhombus consisting of the triangles labelled 0 and 3. Let us denote this probability by q. Any tiling must contain exactly one of these three rhombi, hence we have $p+q+q = p+2q = 1$. Therefore, whenever there is a single sum formula available for p, there is also one for q. To come back to our example, the rhombus whose bottom-most point has coordinates $(2n+2, 2n+1)$ in a hexagon with side lengths $2n+2$, $2n+1$, $2n+1$, $2n+2$, $2n+1$, $2n+1$, can be seen as such a rhombus consisting of triangles labelled 0 and 3, where the bottom-most point of the rhombus consisting of the triangles labelled 0 and 1 has coordinates $(2n+1, 2n+1)$. This puts us in Case (E), with $a' = x' = -1$, (to see this one has to replace n by $n-1$ in the above coordinatization), and thus the claimed expression for $f_{2,1,1,2,1}(n)$ can be proved in the same manner as we proved the expression for $f_{0,0,1,0,0}(n)$ above.

Again, for the sake of completeness, we list the additional vectors (a, b, c, x, y) for which single sums are available for $f_{a,b,c,x,y}(n)$ by the above observation.

(C') By [7, Theorem 1]: $(2a', 0, 0, a'+1, a')$, $(2a', 0, 0, a', a')$, $(2a', 0, 0, a', a'-1)$, $(2a', 0, 0, a'-1, a'-1)$, $(0, 2a', 0, a'+1, 0)$, $(0, 2a', 0, a', -1)$, $(0, 2a', 0, a', 0)$, $(0, 2a', 0, a'-1, -1)$, for an integer a'.

(D') By [7, Theorem 2]: $(2a'+1, 1, 1, a'+2, a'+1)$, $(2a'+1, 1, 1, a'+1, a'+1)$, $(2a'+1, 1, 1, a'+1, a')$, $(2a'+1, 1, 1, a', a')$, $(1, 2a'+1, 1, a'+2, 1)$, $(1, 2a'+1, 1, a'+1, 0)$, $(1, 2a'+1, 1, a'+1, 1)$, $(1, 2a'+1, 1, a', 0)$, for an integer a'.

(E') By [6, Theorems 1 and 2]: $(2a', 0, 0, a'+x', a'-x')$, $(2a', 0, 0, a'+x', a'-x'-1)$, $(2a'+1, 0, 0, a'+x', a'-x')$, $(2a'+1, 0, 0, a'+x', a'-x'+1)$, $(0, 2a', 0, a'+x', 2x'-1)$, $(0, 2a', 0, a'+x', 2x')$, $(0, 2a'+1, 0, a'+x', 2x'-2)$, $(0, 2a'+1, 0, a'+x', 2x'-1)$, for integers a' and x'.

(F') By [6, Theorems 1 and 2]: $(2a', 1, 1, a'+x', a'-x'+1)$, $(2a', 1, 1, a'+x', a'-x')$, $(2a'+1, 1, 1, a'+x', a'-x'+2)$, $(2a'+1, 1, 1, a'+x', a'-x'+1)$, $(1, 2a', 1, a'+x', 2x'-1)$, $(1, 2a', 1, a'+x', 2x')$,

$(1, 2a'+1, 1, a'+x', 2x'-2)$, $(1, 2a'+1, 1, a'+x', 2x'-1)$, for integers a' and x'.

(G') By [7, Theorem 3]: $(2a'+1, 0, 0, a'+2, a'+2)$, $(2a'+1, 0, 0, a'+1, a'+2)$, $(2a'+1, 0, 0, a', a')$, $(2a'+1, 0, 0, a'-1, a')$, $(0, 2a'+1, 0, a'+2, 0)$, $(0, 2a'+1, 0, a'+1, -1)$, $(0, 2a'+1, 0, a', 0)$, $(0, 2a'+1, 0, a'-1, -1)$, for an integer a'.

(H') By [7, Theorem 4]: $(2a', 1, 1, a'+2, a'+2)$, $(2a', 1, 1, a'+1, a'+2)$, $(2a', 1, 1, a', a')$, $(2a', 1, 1, a'-1, a')$, $(1, 2a', 1, a'+2, 1)$, $(1, 2a', 1, a'+1, 0)$, $(1, 2a', 1, a', 1)$, $(1, 2a', 1, a'-1, 0)$, for an integer a'.

(I') By [7, Theorem 5]: $(2a', 0, 0, a'+2, a'+1)$, $(2a', 0, 0, a'+1, a'+1)$, $(2a', 0, 0, a'-1, a')$, $(2a', 0, 0, a'-2, a')$, $(0, 2a', 0, a'+2, 0)$, $(0, 2a', 0, a'+1, -1)$, $(0, 2a', 0, a'-1, 0)$, $(0, 2a', 0, a'-2, -1)$, for an integer a'.

(J') By [7, Theorem 6]: $(2a'+1, 1, 1, a'+3, a'+2)$, $(2a'+1, 1, 1, a'+2, a'+2)$, $(2a'+1, 1, 1, a', a'-1)$, $(2a'+1, 1, 1, a'-1, a'-1)$, $(1, 2a'+1, 1, a'+3, 1)$, $(1, 2a'+1, 1, a'+2, 0)$, $(1, 2a'+1, 1, a', 1)$, $(1, 2a'+1, 1, a'-1, 0)$, for an integer a'.

Table 1 lists the special cases that we considered in (1.2)–(1.4), (1.6), (1.8), (1.10)–(1.12), for which proofs are available, together with an indication from which of the Cases (A)–(H), respectively (C')–(H'), these proofs come from. In particular, the expression for $f_{0,0,1,0,0}(n)$ in (1.11) (together with (1.1)) provides the formula for the values of Pauling's bond order in Tabelle 2 of [11] for higher naphtalenes $N(p)$ of odd order $p = 2n-1$, and the expression for $f_{1,1,0,1,0}(n)$ in (1.8) (together with (1.1)) provides the formula for the values of Pauling's bond order in Tabelle 3 of [11] for higher pyrenes $P(p)$ of odd order $p = 2n-1$. The values in Tabelle 1 of [11] for higher benzenes $B(p)$ are expressed by (1.1) with $a = b = c = x = y = 0$ and $a = b = c = x = y = 1$, respectively, with the expression for $f_{0,0,0,0,0}(n)$ given in (1.10) and the one for $f_{1,1,1,1,1}(n)$ given in (1.4). (The latter formulas have already been stated in [7, Cor. 7, (1.8) and (1.9)]. In fact, Corollary 7 of [7] contains some more evaluations of this kind.)

Coming back to the original goal, a proof of the Conjecture for arbitrary a, b, c, x and y, it may seem that it should be at least possible to achieve this in the Cases (A)–(H) and (C')–(H'), where single sum formulas are available. For, for each specific choice of a, b, c, x and y out of one of these cases, an identity of the form "single sum = closed form" has to be proved. So one would try to follow the strategy that was suggested in complete generality at the beginning of this section: apply some manipulations (using hypergeometric transformation and summation formulas, for example) until the desired expression is obtained. This task is much less daunting here, since we are dealing now with a single sum, not with a triple sum. Moreover, as it turns out, the sums that occur are very familiar objects in hypergeometric theory (we refer the reader to [1, 19, 8] for information on this theory), they turn out to be balanced ${}_4F_3$-series, respectively very-well-poised ${}_7F_6$-series. (For example, the series in (3.1) is a balanced ${}_4F_3$-series.) For these

Table 1

a,	b,	c,	x,	y	entry in (A)–(H), respectively (C')–(H')
-1,	-1,	0,	-1,	-1	(A), $a'=b'=1$
2,	2,	1,	2,	1	(B), $a'=b'=-1$
2,	1,	1,	2,	1	(F'), $a'=x'=1$
2,	1,	1,	1,	1	(F'), $a'=1,\ x'=0$
1,	2,	1,	2,	1	(F'), $a'=x'=1$
1,	2,	1,	1,	0	(F'), $a'=1,\ x'=0$
-1,	0,	0,	0,	-1	(E'), $a'=-1,\ x'=1$
-1,	0,	0,	-1,	-1	(E'), $a'=-1,\ x'=0$
0,	-1,	0,	0,	0	(E'), $a'=-1,\ x'=1$
0,	-1,	0,	-1,	-1	(E'), $a'=-1,\ x'=0$
1,	1,	1,	1,	1	(C), $a'=b'=0$
1,	1,	0,	1,	-1	(H), $a'=b'=0$
1,	1,	0,	1,	1	(H), $a'=b'=0$
1,	1,	1,	0,	-1	(F), $a'=1,\ x'=0$
1,	1,	1,	0,	1	(J'), $a'=0$
1,	1,	1,	1,	0	(C), $a'=b'=0$
0,	2,	0,	1,	0	(E'), $a'=1,\ x'=0$
2,	0,	0,	0,	1	(I'), $a'=1$
2,	0,	0,	1,	0	(E'), $a'=1,\ x'=0$
2,	0,	0,	1,	1	(E'), $a'=1,\ x'=0$
2,	0,	0,	2,	0	(E'), $a'=x'=1$
0,	2,	1,	1,	0	(B), $a'=b'=0$
1,	1,	0,	1,	0	(A), $a'=0,\ b'=1$
0,	0,	0,	0,	0	(D), $a'=b'=0$
0,	0,	1,	0,	0	(B), $a'=b'=1$
3,	3,	0,	3,	1	(A), $a'=b'=0$

series there are a lot of summation and transformation formulas known. However, and this is somehow mysterious, I was not able to establish any of the theorems that I presented here in this classical manner (i.e., without the use of Zeilberger's algorithm), not to mention a general theorem for an infinite family of parameters. As already said at the beginning of this section, the biggest stumbling block in such an attempt is the question of how one would be able to isolate "1/3" from the "rest." So, potentially, there is a hierarchy of interesting hypergeometric identities lurking behind the scene which has not yet been discovered.

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The Influence of Carr's Synopsis on Ramanujan*

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Almost all biographers of Ramanujan (e.g., P.V. Seshu Aiyar and R. Ramachandra Rao [13, p. xii]) point to G.S. Carr's *A Synopsis of Elementary Results in Pure Mathematics* [10] as the book which kindled the fire of Ramanujan's devotion to mathematics. How much did Carr's *Synopsis* influence Ramanujan? Which published papers and which entries in his notebooks [14] have their seeds in the *Synopsis*? We cannot provide definitive answers to these questions, because we know very little about the other books Ramanujan might have studied in his formative years. However, upon a close examination of Carr's book, we can suggest some topics which Ramanujan *might* have learned from Carr.

At the outset, we emphasize that most of Carr's book seems to have had little influence on Ramanujan's development. The majority of subjects expounded by Carr are not evinced in Ramanujan's writings. In particular, the book contains an enormous amount of geometry, a subject to which Ramanujan evidently devoted little of his creative energy. His published work in geometry encompasses one single-page paper on squaring the circle [11], [13, p. 22], two problems submitted to the *Journal of the Indian Mathematical Society* [13, pp. 331, 333], and a short section in his famous paper on modular equations and approximations to π [12, Sect. 12], [13, pp. 35–36], wherein he used a couple of geometrical constructions to find approximations to π. His notebooks contain six geometrical claims, each accompanied by a figure and all found in Chapters 18 and 19 of his second notebook [14], [3, pp. 190, 194, 211, 243, 245, 299].

Ramanujan possibly first learned differential and integral calculus from the *Synopsis*, for it contains considerable material on these subjects, especially on calculating indefinite integrals. Infinite series are not as thoroughly treated as differential and integral calculus.

In the remainder of this article, we indicate some topics in Carr's *Synopsis* which relate to Ramanujan's papers and notebooks. We begin with elementary mathematics.

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Problems submitted by Ramanujan to the *Journal of the Indian Mathematical Society* [13, pp. 327, 329, 334], [9, pp. 18, 20–23] and entries in his notebooks [4, pp. 34–35] indicate Ramanujan's fondness of elementary identities involving radicals. Ramanujan's calculations of class invariants [5, Chap. 34], [7], [8] frequently required elementary radical identities, in particular, denesting theorems. Pages 50–53 in the *Synopsis* treat surds, and, in particular, on page 52 we find the identity

$$\sqrt{a \pm \sqrt{b}} = \sqrt{\tfrac{1}{2}(a + \sqrt{a^2 - b})} \pm \sqrt{\tfrac{1}{2}(a - \sqrt{a^2 - b})},$$

which denests if $a^2 - b$ is a perfect square. Undoubtedly, Ramanujan used this identity numerous times to simplify calculations of class invariants.

Certain results in Ramanujan's notebooks require finding the roots of cubic or quartic polynomials. The Cardan and trigonometric methods for solving cubics are found on pages 127–129, and methods for solving quartics are given on pages 129–130 in Carr's text.

Differential calculus begins on page 257, and integral calculus commences on page 313. The attention given to integrals is considerably more extensive. Pages 343–347 are devoted to elliptic integrals with the elliptic integrals of the first and second kinds, $K(k)$ and $E(k)$, respectively, defined on page 344, and the lemniscate integral appearing on page 347. The integral $K(k)$ is prominent in Ramanujan's extensive work on elliptic functions [3], [5, Chaps. 33, 36], and he proved some beautiful theorems on the lemniscate integral [4, Chap. 26]. However, entries in his notebooks [3, pp. 104–113] suggest that Ramanujan was more fully acquainted with elliptic integrals through a text on elliptic functions. In his *Quarterly Reports* [1, pp. 313–318], Ramanujan established a beautiful generalization of the basic theorem on Frullani integrals. Under suitable hypotheses on f, one version of Frullani's theorem asserts that

$$\int_0^\infty \frac{f(ax) - f(bx)}{x} dx = (f(0) - f(\infty)) \log\left(\frac{b}{a}\right).$$

Also a short fragment published with his lost notebook [15, pp. 274–275], [6] features Frullani integrals. The Frullani integral theorem appears on page 392 in Carr's *Synopsis*, with various examples scattered nearby, for example, on pages 380, 381, and 385.

Some elementary theorems on the convergence and divergence of infinite series appear on pages 80–82. Topics in infinite series are interspersed throughout the book. For example, the partial fraction decompositions of trigonometric functions are found on page 426. Ramanujan was especially fond of partial fraction expansions; see several expansions in Chapters 14, 15, and 18 of his second notebook [2, pp. 237, 248–249, 267–275, 277–279, 291–293, 314–315], [3, pp. 200–206] and scattered decompositions in the unorganized pages of his second and third notebooks [4, Chap. 30]. Ordinary hypergeometric series ${}_2F_1$ are defined on page 97, but with the exception of a continued fraction representation for a certain quotient of hypergeometric functions and two corollaries, no further properties are delineated. Ramanujan rediscovered most of the important classical theorems about ordinary and generalized hypergeometric functions, but found many new theorems in this venerable subject as well [2, Chaps. 10, 11; Chap. 12, pp. 133–145; Chap. 13, pp. 193–205], [5, Chap. 33]. Sections 13–16 in Chapter 3 of Ramanujan's second notebook [1, pp. 69–81] are devoted to expansions derivable from Lagrange's inversion formula, which can be found on page 278 in the *Synopsis*. In fact, one of Ramanujan's examples [1, p. 71], the Maclaurin series

$$\left(\frac{2}{1+\sqrt{1-4t}}\right)^n = 1 + nt + n\sum_{k=2}^{\infty}\frac{\Gamma(n+2k)t^k}{\Gamma(n+k+1)k!},$$

is given in an equivalent form by Carr on page 281. Fourier series, which Ramanujan often used, are introduced on page 432. Throughout his notebooks, Ramanujan made numerous applications of the Euler–Maclaurin summation formula; for example, see [1, pp. 87, 134–136], [2, pp. 300–302]. Most notably, Ramanujan's theory of divergent series and his definition of the "constant" of a series (either convergent or divergent) hinges upon the Euler–Maclaurin formula [1, Chapt 6]. The Euler–Maclaurin formula is given in a very imprecise form in terms of indefinite integrals on pages 558–560 in Carr's *Synopsis*.

It is curious that on page 352 Carr sums two infinite series by setting $n = 1/dx$ in two finite sums of n terms, which can be interpreted as Riemann sums for certain integrals, and then letting dx tend to 0. This peculiar device was also used by Ramanujan in his notebooks [1, p. 27].

Ramanujans continued fraction expansions of analytic functions are among his most beautiful contributions to mathematics. The basic theory of continued fractions is sketched on pages 62–66 in the *Synopsis*. Note that the now archaic terms "definite" and "indefinite" continued fractions for convergent and divergent continued fractions, respectively, are defined on page 63; this terminology was used by Ramanujan [5, p. 35]. Although the elementary theory of continued fractions is adequately presented, there is no material on expansions of functions of either a real or complex variable in Carr's book, except for the aforementioned continued fractions of quotients of hypergeometric functions.

Infinite product representations of a few elementary functions are found on pages 431–432 in Carr's *Synopsis*. One of Ramanujan's papers [13, pp. 50–52] and certain entries in his notebooks [1, pp. 38–41], [2, pp. 230–231, 241], [5, p. 560] focus on representing certain functions by infinite products.

Basic properties of the gamma and beta functions are given on pages 359–364, Stirling's formula for $\log\Gamma(x)$ is on page 412, and considerable material on the logarithmic derivative of $\Gamma(x)$ is located on pages 404–410. The gamma function permeates much of Ramanujan's work. In particular, basic properties are recorded in Chapter 7 of his second notebook [1, pp. 150–180], analogues of the gamma function are studied in Chapters 8 and 9 [1, pp. 196–231, 273–285], and beautiful continued fractions for quotients of gamma functions are found in Chapter 12 [2, pp. 140–164] and the unorganized portions of his notebooks [5, pp. 50–66]. See also his interesting paper on integrals of gamma functions [13, pp. 216–229].

Ramanujan's first published paper [13, pp. 1–14] is on properties of the Bernoulli numbers B_n, $n \geq 0$, which are analytically defined on page 276 in the *Synopsis*. Carr offers a recurrence formula for B_n but does not discuss arithmetical properties of B_n, to which Ramanujan devotes Sections 6–10 in his paper. Euler's formula for $S_{2n} := \zeta(2n) := \sum_{k=1}^{\infty} k^{-2n}$ in terms of B_{2n}, where n is a positive integer, is given on both pages 277 and 430. Formulas for sums related to $\zeta(n)$ are the subject of Ramanujan's fourth published paper [13, pp. 20–21]. Finally, the dilogarithm $\mathrm{Li}_2(x) := \sum_{k=1}^{\infty} x^k/k^2$, $|x| \leq 1$, and some of its basic properties are related on pages 368–370. In Chapter 9 of his second notebook, Ramanujan records several basic properties and values of the dilogarithm $\mathrm{Li}_2(z)$, all of which are classical, for example,

$$\mathrm{Li}_2(1 - z) + \mathrm{Li}_2\left(1 - \frac{1}{z}\right) = -\frac{1}{2}\log^2 z.$$

However, he also examines some analogues of the dilogarithm, and all of these beautiful results are new [1, pp. 246–260].

It has been suggested by several of Ramanujan's biographers that Carr's cryptic style, with lists of theorems followed by sketchy proofs, influenced Ramanujan's habit of successively recording theorems with no proofs in his notebooks. This may be true, but there are perhaps two more important reasons for the lack of proofs in his notebooks. First, Ramanujan worked mainly on his slate, for paper was expensive for him. Thus, only the final formulas or theorems were recorded in his notebooks. Second, Ramanujan probably thought that it was unnecessary for him to record any proofs. The notebooks were his own personal compilation of what he had discovered, and if someone had asked him to prove a certain result on a certain page, he knew he could reproduce his argument.

It also might be mentioned that biographers of Ramanujan usually assert that Carr's *Synopsis* contains about 6000 theorems, since the number assigned to the last entry of the book is 6165. However, the numbering contains many gaps, ranging from a gap of one, of which there are several, to a gap of length 170. The present writer has actually counted the number of articles and found the number to be 4417.

Undoubtedly, we have failed to discern some of the threads leading from Carr's *Synopsis* to Ramanujan's papers and notebooks. On the other hand, because so little is known about Ramanujan's early mathematical education, some of the proposed influences of Carr may be illusory. In summary, Carr's idiosyncratic book most likely had a strong influence on Ramanujan's early work, but as the depth of Ramanujan's discoveries deepened, Carr's influence certainly waned. But we can all be thankful for Carr's spark which ignited the flame.

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A Bailey Lemma from the Quintuple Product

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To the memory of Srinivasa Ramanujan

In a previous paper, the discovery of further Rogers-Ramanujan type identities from new Bailey Lemmas was discussed. In that paper, the starting point was a product of independent Jacobi triple products. In this paper, we start from the quintuple product.

1. Introduction

In his second paper, [6] on Rogers-Ramanujan type identities, L.J. Rogers began with an umbral transformation of Jacobi's triple product. In [4], I used Rogers' idea on a product of several Jacobi triple products. As a result, multi-dimensional Bailey chains (see [3] for the origin of this term) followed naturally, and some new Pentagonal Number Theorems were found. For example, for $|q| < 1$.

$$\sum_{m,n,p\geq 0} \frac{q^{m^2+n^2+p^2}}{(q)_{m+n-p}(q)_{m+p-n}(q)_{p+n-m}}$$

$$= \frac{1}{(q)_\infty^3} \sum_{i,j,k=-\infty}^{\infty} (-1)^{i+j+k} q^{\frac{1}{2}i(3i-1)+\frac{1}{2}j(3j-1)+\frac{1}{2}k(3k-1)+ij+ik+jk}, \tag{1.1}$$

where

$$(a)_n = (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \tag{1.2}$$

and

$$(a)_\infty = \lim_{n\to\infty} (a)_n. \tag{1.3}$$

In the above sum, the indices are restricted so that each of $m+n-p$, $m+p-n$ and $p+n-m$ is non-negative, or for simplicity we may assume $1/(q)_n = 0$ when $n < 0$.

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In this paper, our starting point will be the quintuple product identity [5; p. 134, Ex. 5.6]

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}z^{3n}(1+zq^n)$$
$$= (q)_\infty(-z)_\infty(-q/z)_\infty(qz^2;q^2)_\infty(qz^{-2};q^2)_\infty. \tag{1.4}$$

In Section 2, we shall obtain the Bailey lemmas that this identity implies. In Section 3, we note that the simplest application of our Bailey lemmas is Euler's Pentagonal Number Theorem. In Section 4, we develop backaground necessary for our deeper applications in Section 5. The difficulty arising in this latter application makes clear that more extensive studies of classical very well poised basic hypergeometric series will be necessary for general application of the Quintuple Product Bailey Lemma.

The result we shall prove in Section 5 is

$$\sum_{r,n,t,j\geqq 0}\frac{(-1)^t q^{n^2+n+j(3j+1)/2+r(3r-1)/2+t(11t-1)/2+5rt+5jt+2jr+jn}}{(q)_r(q)_n(q)_t(q)_j}$$
$$(q)_{r+t+j}(1-q^{6r+12t+6j+6})$$
$$=\frac{(q^7;q^7)_\infty(-q^2;q^7)_\infty(-q^5;q^7)_\infty(q^3;q^{14})_\infty(q^{11};q^{14})_\infty}{(q;q^2)_\infty}. \tag{1.5}$$

The right-hand side resembles but is, in fact, quite different from [7; p. 160, eq. (80)].

2. New Bailey Lemmas

Using Jacobi's Triple Product [2; p. 21, eq. (2.2.10)]

$$\sum_{n=-\infty}^{\infty} q^{n^2}y^n = (q^2;q^2)_\infty(-yq;q^2)_\infty(-y^{-1}q;q^2)_\infty, \tag{2.1}$$

we rewrite (1.4) as

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}z^{3n}(1+zq^n)$$
$$=\frac{1}{(-q;q)_\infty}(-z)_\infty(-q/z)_\infty\sum_{r=-\infty}^{\infty}(-1)^r z^{2r}q^{r^2}. \tag{2.2}$$

Next we expand the two products in the numerator of (2.2) using Euler's identity [2; p. 19, eq. (2.2.6)]

$$\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}y^n}{(q)_n}=(-y)_\infty. \tag{2.3}$$

Hence

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}z^{3n}(1+zq^n)$$
$$=\frac{1}{(-q;q)_\infty}\sum_{i=0}^{\infty}\frac{q^{\binom{i}{2}}z^i}{(q)_i}\sum_{j=0}^{\infty}\frac{q^{\binom{j+1}{2}}z^{-j}}{(q)_j}\sum_{r=-\infty}^{\infty}(-1)^r z^{2r}q^{r^2}. \tag{2.4}$$

Following the original ideas of Rogers, we replace all odd powers of z by 0 and all even powers z^{2n} by α_n; furthermore in the i and j sum we replace i by $n+h$ and j by $n-h$ (which is admissable because only even power of z are not annihilated).

Hence we obtain (subject to obvious convergence conditions):

Lemma 1.

$$\sum_{n=-\infty}^{\infty}q^{n(6n-1)}\alpha_{3n}-\sum_{n=-\infty}^{\infty}q^{(2n-1)(3n-1)}\alpha_{3n-1}$$
$$=\frac{1}{(-q;q)_\infty}\sum_{n\geq 0}\sum_{r=-\infty}^{\infty}(-1)^r q^{r^2+n^2}\beta_n(r), \tag{2.5}$$

where

$$\beta_n(r)=\sum_{h=-n}^{n}\frac{q^{h^2-h}\alpha_{h+r}}{(q)_{n-h}\,(q)_{n+h}}. \tag{2.6}$$

On the other hand, we may replace all even powers of z in (2.4) by 0 and all odd powers z^{2n-1} by α_n; this time $i=n+h-1$ and $j=n-h$.

We now obtain (upon replacing n by $n+1$):

Lemma 2.

$$\sum_{n=-\infty}^{\infty}q^{n(6n+1)}\alpha_{3n+1}-\sum_{n=-\infty}^{\infty}q^{(2n-1)(3n-2)}\alpha_{3n-1}$$
$$=\frac{1}{(-q;q)_\infty}\sum_{n\geq 0}\sum_{r=-\infty}^{\infty}q^{r^2+n^2+n}(-1)^r\overline{\beta}_n(r), \tag{2.7}$$

where

$$\overline{\beta}_n(r)=\sum_{h=-n-1}^{n}\frac{q^{h^2}\alpha_{h+r+1}}{(q)_{n-h}\,(q)_{n+h+1}}. \tag{2.8}$$

3. Euler's Pentagonal Number Theorem

Let us examine Lemma 1, when $\alpha_n = (-1)^n$. In this case, the left-hand side of (2.5) becomes

$$\begin{aligned} &\sum_{n=-\infty}^{\infty} q^{n(6n-1)}(-1)^n + \sum_{n=-\infty}^{\infty} (-1)^n q^{6n^2-5n+1} \\ (3.1) \qquad &= \sum_{n=-\infty}^{\infty} (-1)^n (-q)^{n(3n-1)/2}. \end{aligned}$$

Putting $\alpha_{h+r} = (-1)^{h+r}$ in (2.6), we find by [5; p. 238, (II.21)]

$$(3.2) \qquad \beta_n(r) = \frac{q^n(-1)^r}{(q^2;q^2)_n}$$

Hence under the substitution $\alpha_n = (-1)^n$, the right-hand side of (2.5) becomes

$$\begin{aligned} &\frac{1}{(-q;q)_\infty} \sum_{n\geq 0} \sum_{r=-\infty}^{\infty} \frac{q^{r^2+n^2+n}}{(q^2;q^2)_n} \\ &= \frac{1}{(-q;q)_\infty} (q^2;q^2)_\infty (-q;q^2)_\infty^2 (-q^2;q^2)_\infty \\ &\qquad\qquad\qquad \text{(by (2.1) and (2.3))} \\ (3.3) \qquad &= (-q;-q)_\infty. \end{aligned}$$

Matching up (3.1) and (3.3) and changing q into $-q$, we find Euler's Pentagonal Number Theorem [2; p. 11, eq. (1.3.1)]

$$(3.4) \qquad (q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}.$$

Of course, this provides no compelling reason to study Lemmas 1 or 2. Indeed, Euler's Pentagonal Number Theorem is, in fact, a Corollary of Jacobi's Triple Product itself. Consequently, we must await Section 5 to see really new identities following from the Lemmas in Section 2.

4. Background from Very Well-Poised Series

We require for Section 5, new representations of some of the series appearing in [1; p. 433]. In particular,

$$(4.1) \qquad C_{1,r}(a;x;q) = \sum_{n\geq 0} x^n a^{-n} q^{n^2+n-rn} \frac{(1-x^r q^{2nr})(x)_n (a)_n}{(1-x)(q)_n \left(\frac{xq}{a}\right)_n},$$

and

$$H_{1,r}(a;x;q) = \frac{(\frac{xq}{a})_\infty}{(xq)_\infty} C_{1,r}(a;x;q). \tag{4.2}$$

From [1; p. 439, eq. (3.7)], we see that

$$H_{1,1}(a;x;q) = 1, \tag{4.3}$$

and from [1; p. 439, eq. (3.4)], we deduce that

$$H_{1,2}(a;xq;q) = 1 + \frac{xq}{a}. \tag{4.4}$$

From [1; p. 1435, eqs. (2.1) with $k = \lambda = 1, i = 2 - r$],

$$J_{1,r}(a;x;q) = H_{1,r-1}(a;x;q) - x\, H_{1,r-2}(a;x;q), \tag{4.5}$$

where we have invoked [1; p. 435, eq. (2.3)]. The $J_{1,r}(a;x;q)$ is defined in [1; p. 433, eq. (1.4)], but it will be eliminated shortly. So its exact definition is not required here. Also from [1; p. 435, eq. (2.2)]

$$J_{1,r}(a;x;q) = H_{1,r}(a;xq;q) - \frac{xq}{a} H_{1,r-1}(a;xq;q). \tag{4.6}$$

Eliminating $J_{1,r}(a;x;q)$ from (4.5) and (4.6), we find

$$H_{1,r}(a;xq;q) = \frac{xq}{a} H_{1,r-1}(a;xq;q) + H_{1,r-1}(a;x;q) - x H_{1,r-2}(a;x;q). \tag{4.7}$$

Equations (4.3), (4.4) and (4.7) reveal that the $H_{1,r}(a;x;q)$ are polynomials in x, a^{-1}, q and q^{-1}.

Our final result in this section provides an explicit representation for these polynomials.

Lemma 3. *For each integer* $r \geqq 1$,

$$\begin{aligned} &H_{1,r}(a; xq^{r-1}; q) \\ (4.8)\qquad &= \sum_{r-1\geqq t,j\geqq 0} \begin{bmatrix} r-t-1 \\ t, j, r-2t-j-1 \end{bmatrix} (-1)^t x^{t+j} a^{-j} q^{(t+j)^2+\binom{t}{2}} \end{aligned}$$

where $\begin{bmatrix} A \\ B, C, A-B-C \end{bmatrix} = (q;q)_A/((q;q)_B(q;q)_C(q;q)_{A-B-C})$

Proof. Let us designate the right-hand side of (4.8) as $h_r(xq^{r-1})$. By inspection we see that

$$h_1(x) = 1, \tag{4.9}$$

and

$$h_2(xq) = 1 + \frac{xq}{a}, \tag{4.10}$$

Next

$$\begin{aligned} &\frac{xq^{r-1}}{a} h_{r-1}(xq^{r-1}) + h_{r-1}(xq^{r-2}) - xq^{r-2} h_{r-2}(xq^{r-2}) \\ &= \sum_{t,j\geqq 0} \begin{bmatrix} r-t-2 \\ t, j, r-2t-j-2 \end{bmatrix} (-1)^t x^{t+j+1} a^{-j-1} q^{(t+j)^2+\binom{t}{2}+t+j+r-1} \\ &+ \sum_{t,j\geqq 0} \begin{bmatrix} r-t-2 \\ t, j, r-2t-j-2 \end{bmatrix} (-1)^t x^{t+j} a^{-j} q^{(t+j)^2+\binom{t}{2}} \\ &- \sum_{t,j\geqq 0} \begin{bmatrix} r-t-3 \\ t, j, r-2t-j-3 \end{bmatrix} (-1)^t x^{t+j+1} a^{-j} q^{(t+j)^2+\binom{t}{2}+t+j+r-2} \\ &= \sum_{t,j\geqq 0} \begin{bmatrix} r-t-2 \\ t, j-1, r-2t-j-1 \end{bmatrix} (-1)^t x^{t+j} a^{-j} q^{(t+j-1)^2+\binom{t}{2}+t+j+r-2} \\ &+ \sum_{t,j\geqq 0} \begin{bmatrix} r-t-2 \\ t, j, r-2t-j-2 \end{bmatrix} (-1)^t x^{t+j} a^{-j} q^{(t+j)^2+\binom{t}{2}} \\ &+ \sum_{t,j\geqq 0} \begin{bmatrix} r-t-2 \\ t-1, j, r-2t-j-1 \end{bmatrix} (-1)^t x^{t+j} a^{-j} q^{(t+j-1)^2+\binom{t-1}{2}+t+j+r-3} \end{aligned}$$

$$= \sum_{t,j\geqq 0} \frac{(q;q)_{r-t-2}}{(q;q)_t (q;q)_j (q;q)_{r-2t-j-1}} (-1)^t x^{t+j} a^{-j} q^{(t+j)^2+\binom{t}{2}}$$

$$\times\{(1-q^j)q^{-2t-2j+1+t+j+r-2}+(1-q^{r-2t-j-1})$$
$$+(1-q^t)q^{-2t-2j+1+j+r-2}\}$$
$$=\sum_{t,j\geqq 0}\frac{(q;q)_{r-t-1}(-1)^t x^{t+j}a^{-j}q^{(t+j)^2+\binom{t}{2}}}{(q;q)_t(q;q)_j(q;q)_{r-2t-j-1}}$$
$$=h_r(xq^{r-1}). \tag{4.11}$$

So by (4.9) and (4.10) $h_r(xq^{r-1})$ satisfies the same initial conditions as $H_r(a;xq^{r-1};q)$ (see (4.3) and (4.4)).

In addition by (4.11) and (4.7), we see that $h_r(xq^{r-1})$ and $H_{1,r}(a;xq^{r-1};q)$ satisfy the same second order recurrence. Therefore

$$H_{1,r}(a;xq^{r-1};q)=h_r(xq^{r-1}), \tag{4.12}$$

and Lemma 3 is proved. □

Corollary. *For* $r\geqq 1$.

$$C_{1,r}(a;x;q)=\frac{(xq)_\infty}{\left(\frac{xq}{a}\right)_\infty}$$
$$\sum_{r-1\geqq t,j\geqq 0}\begin{bmatrix} r-t-1 \\ t,j,r-2t-j-1\end{bmatrix}(-1)^t x^{t+j}a^{-j}q^{(t+j)^2+\binom{t}{2}+(1-r)(t+j)}$$

Proof. This follows immediately from Lemma 3 and (4.2). □

5. Proof of Identity (1.5)

We shall utilize Lemma 2 in the instance

$$\alpha_n=(-1)^n q^{\binom{n}{2}}. \tag{5.1}$$

Hence in this case,

$$\overline{\beta}_n(r)=\sum_{h=0}^{n}\frac{q^{h^2}(-1)^{h+r+1}q^{\binom{h+r+1}{2}}}{(q)_{n-h}(q)_{n+h+1}}$$
$$+\sum_{h=0}^{n}\frac{q^{(-h-1)^2}(-1)^{h+r}q^{\binom{-h+r}{2}}}{(q)_{n-h}(q)_{n+h+1}}$$
$$=(-1)^{r+1}q^{\binom{n+1}{2}}\sum_{h=0}^{n}\frac{q^{h^2+\binom{h+1}{2}+hr}(-1)^h}{(q)_{n-h}(q)_{n+h+1}}(1-q^{2h+1-2hr-r})$$

$$= \frac{(-1)^{r+1} q^{\binom{r+1}{2}}}{(q)_n (q)_{n+1}} \sum_{h=0}^{n} \frac{(q^{-n})_h}{(q^{n+2})_h} q^{h^2+h+h(r+n)} (1 - q^{(2h+1)(1-r)})$$

$$= \frac{(-1)^{r+1} q^{\binom{r+1}{2}}}{(q)_n (q^2; q)_n} C_{1,1-r}(q^{-n}; q; q)$$

$$= \frac{(-1)^{r} q^{\binom{r}{2}+1}}{(q)_n (q^2; q)_n} C_{1,r-1}(q^{-n}; q; q). \tag{5.2}$$

Next we consider the left-hand side of (2.7) under the substitution (5.1). The result is

$$\sum_{n=-\infty}^{\infty} q^{n(6n+1)} (-1)^{3n+1} q^{\binom{3n+1}{2}} - \sum_{n=-\infty}^{\infty} q^{6n^2-7n+2} (-1)^{3n-1} q^{\binom{3n-1}{2}}$$

$$= - \sum_{n=-\infty}^{\infty} q^{\frac{21}{2}n^2+\frac{5n}{2}} (-1)^n + \sum_{n=-\infty}^{\infty} q^{\frac{21n^2}{2}-\frac{23}{2}n+3} (-1)^n$$

$$= - \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{21}{2}n^2+\frac{5n}{2}} - \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{21}{2}n^2+\frac{19n}{2}+2}$$

$$= -(q^7; q^7)_\infty (-q^2; q^7)_\infty (-q^5; q^7)_\infty (q^3; q^{14})_\infty (q^{11}; q^{14})_\infty$$

(by (1.4)).

As for the right-hand side of (2.7), we find that it is (noting that $\overline{\beta}_n(1) = 0$ by (5.2))

$$\frac{1}{(-q; q)_\infty} \left\{ \sum_{n \geqq 0} \sum_{r=2}^{\infty} q^{r^2+n^2+n} (-1)^r \overline{\beta}_n(r) \right.$$

$$\left. + \sum_{n \geq 0} \sum_{r=0}^{\infty} q^{r^2+n^2+n} (-1)^r \overline{\beta}_n(-r) \right\}$$

$$= \frac{1}{(-q; q)_\infty} \sum_{n \geqq 0} q^{n^2+n} \sum_{r=0}^{\infty} (-1)^r (q^{4r+r} \overline{\beta}_n(r+2) + \overline{\beta}_n(-r))$$

$$= \frac{1}{(-q; q)_\infty} \sum_{n \geqq 0} \frac{q^{n^2+n}}{(q)_n (q^2; q)_n}$$

$$\sum_{r=0}^{\infty} q^{r^2}(-1)^r((-1)^r q^{\binom{r+2}{2}+4r+5} C_{1,r+1}(q^{-n};q;q)$$

$$-(-1)^r q^{\binom{r}{2}} C_{1,r+1}(q^{-n};q;q)$$

$$= \frac{-1}{(-q;q)_\infty} \sum_{n\geqq 0} \frac{q^{n^2+n}}{(q)_n(q^2;q)_n}$$

$$\sum_{r=0}^{\infty} q^{r(3r-1)/2} C_{1,r+1}(q^{-n};q;q) \times (1-q^{6r+6})$$

$$= \frac{-1}{(-q;q)_\infty} \sum_{n\geqq 0} \frac{q^{n^2+n}}{(q)_n(q^2;q)_n} \sum_{r=0}^{\infty} q^{r(3r-1)/2}(1-q^{6r+6})$$

$$(q^2;q)_n \sum_{0\leqq t,j\leqq r} \begin{bmatrix} r-t \\ t,j,r-2t-j \end{bmatrix} (-1)^t q^{nj}$$

$$\times\, q^{(t+j)^2+\binom{t}{2}+(t+j)(1-r)}$$

(by Lemma 3)

$$= \frac{-1}{(-q;q)_\infty} \sum_{r,n,t,j\geqq 0} (-1)^t$$

$$\frac{q^{n^2+n+j(3j+1)/2+r(3r-1)/2+t(11t-1)/2+5rt+5jt+2jr+jn}}{(q)_n(q)_t(q)_j(q)_r}$$

$$(q)_{r+t+j}(1-q^{4r+12t+6j+6})$$

which is identity (1.5), as desired.

6. Conclusion

The essential role of Lemma 3 in the proof of (1.5) indicates clearly the difficulties entailed in the applications of either Lemma 1 or Lemma 2. Indeed, in order to apply these lemmas, it seems likely that one will at a minimum require analogs of Lemma 3 for the $C_{k,i}((a);x;q)_\lambda$ of [1; p. 433, eq. (1.1)] with i an arbitrary positive integer.

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Little Flowers to G.H. Hardy
(07–02–1877– 01–12–1947)

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Honouring Ramanujan is not complete without honouring G.H. Hardy who collaborated with him in an epoch-making way and brought his contributions to the lime light of the world. In this small article I list a few results of mine and offer it to G.H. Hardy as little flowers.

1. Introduction

There are many articles (by various authors) regarding the originality of Srinivasa Ramanujan and the great help given to him by G.H. Hardy to see that the works of Ramanujan sees the lime light of the world. In particular the epoch-making results contained in the glorious collaboration of the two great mathematicians Hardy and Ramanujan has been narrated by me in my article [KR]. I will not repeat any of these except in remark below Flower 1. In the present article I list a few results of mine and offer it (as little flowers) to Professor G.H. Hardy.

2. Little Flowers

Flower 1. *Let* $N \geq C$, *where* $C > 0$ *is a certain explicit constant and* N *any odd integer. Let* $R(N)$ *denote the number of triplets* (P_1, P_2, P_3) *of odd primes* P_1, P_2, P_3 *which add up to* N. *Then*

$$R(N) = M(N) + E(N) \tag{2.1}$$

where

$$M(N) = \frac{1}{2} N^2 (\log N)^{-3} \prod_{p} (1 + (p-1)^{-3})$$

$$\prod_{p/N} (1 - (p^2 - 3p + 3)^{-1}) \tag{2.2}$$

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where the first product runs over all primes (and with the condition p/N in the second product),

$$|E(N)| \leq C(\varepsilon)N^2(\log N)^{-4+\varepsilon} \tag{2.3}$$

$\varepsilon > 0$ *being arbitrary and* $C(\varepsilon)$ *depends only on* ε *and is explicit.*

Remarks. I start with the grand history of this result. The origin of this result is that the coefficient of x^n in $(\Sigma_{n=-\infty}^{\infty}(-x)^{n^2})^{-1}$ is the integer nearest to

$$\frac{1}{4n}\left(\cosh(\pi\sqrt{n}) - \frac{\sinh(\pi\sqrt{n})}{\pi\sqrt{n}}\right). \tag{2.4}$$

This result is due to Ramanujan (Ref: his letter dated 16-01-1913 to Hardy written from Madras Port Trust; vide page xxvii of [SR] and also equation (1.14) of [GHH]). Only a great expert like Hardy could realise that this is possibly only an approximation (others perhaps will throw this result to the dust-bin saying that this is false). We do not know how Ramanujan obtained this result. But suffice to say that they (Hardy and Ramanujan) discovered a deep extraordinary method (christened as CIRCLE METHOD by later workers) to study the coefficient of x^n in

$$\prod_{n=1}^{\infty}(1 - x^n)^{-1}. \tag{2.5}$$

Their epoch-making paper opened the flood gates of very important deep discoveries by many great mathematicians all over the world. One of these results is a joint paper by G.H. Hardy and J.E. Littlewood where they studied coefficent of x^n in

$$\left(\sum_p x^p\right)^3 \tag{2.6}$$

where p runs over odd primes. They arrived at a formula like (2.1), where $M(N)$ is a suitable modification of that in Flower 1 and found that the corresponding $E(N)$ satisfies

$$|E(N)| < C(\varepsilon)N^{\frac{7}{4}+\varepsilon}. \tag{2.7}$$

In their proof of (2.7) they assumed something called "Generalised Riemann Hypothesis" which is an unproved hypothesis. The net result was that every odd integer from a certain point onwards can be expressed as a sum of 3 odd primes (provided we assume GRH). A great paper in this direction is a paper by the Soviet Mathematician I.M. Vinogradov who arrived at (2.1) with the inequality

$$|E(N)| < C(\varepsilon)N^2(\log N)^{-\frac{7}{2}+\varepsilon}.$$

It is very important to note that he does not use any unproved hypothesis like GRH. Myself and A. Sankaranarayan consider ourselves very fortunate to have proved Flower 1 by some slight changes (like Vaughan's identity) in the fundamental paper of I.M. Vinogradov. For references to all these results see $[KR, AS]_1$. The referee has kindly drawn my attention to the following work A.A. Karatsuba namely (Basic analytic number theory, Springer-Verlag, Berlin, 1993) where he proves a slightly better result than (2.3) namely the one where ε is replaced by zero and $C(\varepsilon)$ by an absolute constant.

In what follows $s = \sigma + it$ and we denote by T any real number which will exceed a suitable positive constant depending on other parameters.

Flower 2. *We have*

$$\frac{1}{H}\int_{T}^{T+H}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2} dt < D_2(\log T)^{100} \tag{2.8}$$

where $H = T^{\frac{1}{3}}$ *and* $D_2 > 0$ *is a constant.*

Remark. This is a special case of a more general result due to myself and R. Balasubramanian (see $[RB, KR]_1$). The important thing is that (2.8) can be proved by using only three simple things. First the functional equation, next the multiple averaging process and finally Lemma 5.9 of [ECT],

Flower 3. *We have*

$$\min_{T\leq t\leq T+T^{\frac{1}{3}}} \max_{\sigma} |\zeta(\sigma+it)|^{-1} < \exp(D_1(\log\log T)^2) \tag{2.9}$$

where $D_1 > 0$ *is a constant.*

Remark. This result is due to myself and A. Sankaranarayanan (see $[KR, AS]_2$).

Flower 4. *There exists a constant* $\delta > 0$ *with the following property. The number of zeros of* $\zeta'(s)$ *in* $\sigma \geq \frac{1}{2} + \delta$, $T \leq t \leq T + H$ *exceeds* $D_1(\varepsilon)H$, *provided* $H = T^{\varepsilon}$ *where* $\varepsilon > 0$ *is any arbitrary constant and* $D_1(\varepsilon) > 0$ *depends on* ε.

Remark. This result (a special case of a more general result due to myself and R. Balasubramanian, see $[RB, KR]_2$) is still true if $\zeta'(s)$ is replaced by $\zeta'(s) - a$ where a is any complex constant. In that case $\delta > 0$ depends on a.

Flower 5. *Let* $\delta > 0$ *and* $\varepsilon > 0$ *be any two constants and let a be any nonzero complex constant. Then the number of zeros of* $\zeta(s) - a$ *in* $\sigma \geq 1 - \delta$, $T \leq t \leq T + T^{\varepsilon}$ *exceeds* $D_2(\varepsilon)T^{\varepsilon}$ *where* $D_2(\varepsilon) > 0$ *depends only on* δ, ε *and* a.

Remark. This result is the special case of a more general result due to myself and R. Balasubramanian (see $[RB, KR]_3$).

Flower 6. *Let* $k \geq 2$ *be any integer and* $\alpha_1, \alpha_2, \alpha_3, \ldots$ *be any sequence of real numbers each of which is of absolute value* $\leq \frac{1}{3}$. *Then the number of zeros of the function*

$$\sum_{n=1}^{\infty} d_k(n)((n+\alpha_n)^{-s} - n^{-s}) + \zeta^k(s) \tag{2.10}$$

in $|\sigma - \frac{1}{2}| \leq \varepsilon$, $T \leq t \leq t \leq 2T$ *exceeds* $D_3(k,\varepsilon)T \log T$. *Here* $\varepsilon > 0$ *is any arbitary constant and* $D_3(k,\varepsilon) > 0$ *depends only on* k *and* ε. *Also* $d_k(n)$ *are defined by*

$$\zeta^k(s) = \sum_{n=1}^{\infty} d_k(n) n^{-s}.$$

Remark. This is a special case of a more general result due to R. Balasubramanian, myself and A. Sankaranarayanan (see [RB, KR, AS]). In particular see Theorems 8 and 9 of that reference.

Flower 7. *Let* $k \geq 3$ *be any integer and* $Q(x_1, \ldots, x_k)$ *any positive definite quadratic form in* k *variables* $x_1, x_2, \ldots, x_k$ *with integer coefficients. Put*

$$F(s) = \sum (Q(n_1, n_2, \ldots, n_k))^{-s} \tag{2.11}$$

where the sum is over all k *tuples of integers* $n_1, \ldots, n_k$ *with the exception of* $(0,0,\ldots,0)$. *Let* $\varepsilon > 0$ *be any arbitary constant. Then the number of zeros of* $F(s)$ *in* $(|\sigma - \frac{1}{2}| \leq \varepsilon, T \leq t \leq 2T)$ *exceeds* $D_4(\varepsilon)T \log T$ *where* $D_4(\varepsilon)$ *depends only on* ε *and the coefficients of the quadratic form.*

Remark. It is well-known (see for example "Lectures on advanced analytic number theory by C.L. Siegel, T.I.F.R. (Bombay) 1961, pages 60–73) that if $k \geq 2$ then $F(s)$ is a meromorphic function with only one simple pole namely at $s = k/2$. Also it is well known to experts that if $k = 2$ then the number of zeros is question is asymptotic to $C_5 T \log T$ where C_5 depends only on the coefficients of the quadratic form (the proof of this last statement is very much similar to that of Theorem 9.4 [ECT]. We have to use the functional equation which is proved in the lectures of C.L. Siegel mentioned above). Flower 7 is due to myself and A. Sankaranarayan (see $[KR, AS]_3$).

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Rogers-Ramanujan Type Identities for Burge's Restricted Partition Pairs Via Restricted Frobenius Partitions

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We obtain generating functions for two sets of infinite families of restricted Frobenius partition functions by giving combinatorial arguments. We also establish a connection between three particular cases of these restricted Frobenius partition functions and Burge's restricted partition pairs (J. Combin. Theory Ser. A, 63, 1993, 210–222). This connection and Burge's Theorem 1 give us three new analytic identities. A comparison of these analytic identities with three known identities from Slater's compendium (Proc. London Math. Soc. (2). 54, 1952, 147–167) leads us to Rogers-Ramanujan type identities for Burge's restricted partition pairs.

1. Introduction

A Frobenius partition [3] of a positive integer n is a two-rowed array

$$\begin{pmatrix} a_1 \ldots a_m \\ b_1 \ldots b_m \end{pmatrix} \tag{1.1}$$

of non-negative integers arranged in the strictly decreasing order such that

$$n = m + \sum_{i=1}^{m} (a_i + b_i). \tag{1.2}$$

For an odd integer $k \geq -1$, let $A_k(n)$ and $B_k(n)$ denote the number of Frobenius partitions of n which for $1 \leq i \leq m$ satisfy the conditions.

$$a_i \geq b_i \text{ and } b_i \geq \left(\frac{k+3}{2}\right) + a_{i+1} \tag{1.3}$$

and

$$a_i > b_i \text{ and } b_i \geq \left(\frac{k+1}{2}\right) + a_{i+1}, \tag{1.4}$$

respectively.

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Next we recall the definition of Burge's restricted partition pairs from [2]

Definition. (W.H. Burge) A partition with no more than N parts whose largest part is not greater than M is called a partition in (N, M) or an (N, M)-partition. Let $p(N, M, a, b, \alpha, \beta, n)$ be the number of pairs of partitions with total sum n, namely $p_N \geq p_{N-1} \geq \cdots \geq p_1$ and $q_N \geq q_{N-1} \geq \cdots \geq q_1$ both in (N, M) and possibly containing zeros. a, b, α, β are non-negative and

$$p_i - q_{i+1-a} \geq 1 - \alpha, \tag{1.5}$$

$$q_i - p_{i+1-b} \geq 1 - \beta. \tag{1.6}$$

In order to define comparison outside the range, both partitions will be extended with parts M at the high end and zeros at the low end.

We shall use the following notations.

$$P(N, M, a, b, \alpha, \beta, q) = \sum_{n \geq o} p(N, M, a, b, \alpha, \beta, n) q^n, \tag{1.7}$$

$$C_1(n) = \sum_{N=o}^{n} \lim_{M \to \infty} p(N, M, 1, 2, 1, 0, n - N), \tag{1.8}$$

$$C_2(n) = \sum_{N=o}^{n} \lim_{M \to \infty} p(N, M, 1, 2, 0, 1, n - N), \tag{1.9}$$

$$C_3(n) = \sum_{N=o}^{n} \lim_{M \to \infty} p(N, M, 1, 2, 0, 0, n - N). \tag{1.10}$$

In our work we shall use Gaussian polynomials defined by

$$\begin{bmatrix} N+M \\ N \end{bmatrix} = \frac{(q;q)_{N+M}}{(q;q)_N \, (q;q)_M}, \tag{1.11}$$

where $(q;q)_n = \prod_{i=1}^{\infty} \frac{(1-q^i)}{(1-q^{n+i})}$.

We shall also require the following three identities from Slater's compendium [4].

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_{2n}} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} (1 - q^{10n-2})(1 - q^{10n-8})(1 - q^{20n-14}) \times (1 - q^{20n-6})(1 - q^{10n}),$$

(1.12) [4, (79)–(98)]

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_{2n}} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} (1 - q^{10n-1})(1 - q^{10n-9})(1 - q^{20n-8}) \times (1 - q^{20n-12})(1 - q^{10n}),$$

(1.13) [4, (99)]

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q;q)_{2n}} = \frac{1}{(q;q)_\infty}\prod_{n=1}^{\infty}(1-q^{8n-1})(1-q^{8n-7})(1-q^{16n-10})$$
$$\times(1-q^{16n-6})(1-q^{8n}).$$
(1.14) [4, (39)–(83)]

In our next section we obtain generating functions for $A_k(n)$ and $B_k(n)$. In Section 3 we show a connection between $A_{-1}(n)$ and $C_1(n)$, $B_{-1}(n)$ and $C_2(n)$ and $B_1(n)$ and $C_3(n)$ and use this connection and Burge's Theorem 1 to obtain new analytic identities.

Using identities (1.12)–(1.14), we in Section 4 state Rogers-Ramanujan type identities on MacMahon's lines for the functions C_i(n),$1 \leq i \leq 3$. In Section 5 we give the finite version of our new analytic identities (3.5)–(3.7). We conclude in Section 6 by posing an open problem.

2. Generating Functions for $A_k(n)$ and $B_k(n)$

Our object is to prove that

$$\sum_{n=0}^{\infty} A_k(n)\,q^n = \sum_{n=0}^{\infty}\frac{q^{[(k+3)n(n-1)]/2+n}}{(q;q)_{2n}}, \tag{2.1}$$

and

$$\sum_{n=0}^{\infty} B_k(n)\,q^n = \sum_{n=0}^{\infty}\frac{q^{[(k+3)n(n-1)]/2+2n}}{(q;q)_{2n}}. \tag{2.2}$$

Each of these generating functions is proved in a similar way. For (2.1) we provide the details while for (2.2) we give only the q-functional equation.

Proof of (2.1). Let $A_k(m,n)$ denote the number of Frobenius partitions of n enumerated by $A_k(n)$ with the added restriction that there be exactly m columns. We split the partitions enumerated by A_k (m,n) into three classes as follows:

(i) those that have mth column $= \binom{a}{b}$, $b \geq 1$

(ii) those that have mth column $= \binom{0}{0}$,

(iii) those that have mth column $= \binom{a}{0}$, $a \geq 1$.

First, we transform the partitions in class (i) by subtracting 1 from each of the entries of the top and bottom rows. This results in a partition of $n-2m$ into m columns. Further the inequatities in (1.3) are not disturbed by this transformation. Thus the transformed partitions are of the type enumerated by $A_k(m, n-2m)$.

Next we transform the partitions of class (ii) by deleting the mth column and then subtracting $\frac{k+3}{2}$ from each of the remaining entries of the top and bottom rows. This produces a Frobenius partition of $n-(k+3)m+k+2$ into $m-1$ columns. This transformation also does not disturb the inequalities in (1.3). Hence the transformed partitions are of the type enumerated by $A_k(m-1, n-(k+3)m+(k+2))$.

Finally, we transform the partitions of class (iii) by subtracting 1 from each entry of the top row and 1 from each entry of the bottom row except the last. This produces a Frobenius partition enumerated by $A_k(m, n-2m+1)$ in which the mth column is $\binom{a}{0}$, $a \geq 0$. Thus the actual number of partitions in class (iii) is obtained by subtracting from $A_k(m, n-2m+1)$ the number of those partitions which are enumerated by $A_k(m, n-2m+1)$ with last column $\binom{a}{b}$, $b \geq 1$ which is equal to $A_k(m, n-4m+1)$ by case (i). Clearly the number of partitions in class (iii) is $A_k(m, n-2m+1) - A_k(m, n-4m+1)$.

Thus we get the following recurrence formula for $A_k(m,n)$

$$\begin{aligned} A_k(m,n) &= A_k(m, n-2m) + A_k(m-1, n-(k+3)m+k+2) \\ &\quad + A_k(m, n-2m+1) - A_k(m, n-4m+1). \end{aligned} \tag{2.3}$$

$$\text{Let } f_k(z,q) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} A_k(m,n)\, z^m q^n. \tag{2.4}$$

Substituting for $A_k(m,n)$ from (2.3) in (2.4) and then simplifying we get

$$\begin{aligned} \text{Let } f_k\,(z,q) &= f_k(zq^2,q) + zq\, f_k(zq^{k+3},q) \\ &\quad + q^{-1} f_k(zq^2,q) - q^{-1} f_k(zq^4,q). \end{aligned} \tag{2.5}$$

$$\text{Let } f_k(z,q) = \sum_{n=0}^{\infty} \alpha_n(q) z^n \text{ with } \alpha_0(q) = 1. \tag{2.6}$$

Comparing the coefficients of z^n in (2.5), we get

$$\alpha_n(q) = \frac{q^{(k+3)(n-1)+1}}{(1-q^{2n})(1-q^{2n-1})} \alpha_{n-1}(q). \tag{2.7}$$

Iterating (2.7) n times and observing that $\alpha_0(q) = 1$, we obtain

$$\alpha_n(q) = \frac{q^{(k+3)n(n-1)/2+n}}{(q;q)_{2n}}. \tag{2.8}$$

Hence

$$f_k(z,q) = \sum_{n=o}^{\infty} \frac{q^{(k+3)n(n-1)/2+n}}{(q;q)_{2n}} z^n. \tag{2.9}$$

Consider

$$\begin{aligned}\sum_{n=0}^{\infty} A_k(n)\, q^n &= \sum_{n=o}^{\infty}\left[\sum_{m=o}^{\infty} A_k(m,n)\right] q^n \\ &= f_k(1,q) \\ &= \sum_{n=o}^{\infty} \frac{q^{(k+3)n(n-1)/2+n}}{(q;q)_{2n}}.\end{aligned}$$

This completes the proof of (2.1).

In the proof of (2.2) we have used the following q-functional equation

$$\begin{aligned}g_k(z,q) &= zq^2\, g_k(zq^{k+3},q) + g_k(zq^2,q) \\ &\quad + q^{-1}\, g_k(zq^2,q) - q^{-1} g_k(zq^4,q),\end{aligned} \tag{2.10}$$

where

$$g_k(z,q) = \sum_{n=o}^{\infty} \frac{q^{(k+3)n(n-1)/2+2n}}{(q;q)_{2n}} z^n.$$

3. New Analytic Identities

Under the transformation

$$p_i = a_{N-i+1} \text{ and } q_i = b_{N-i+1},\ 1 \le i \le N,$$

where N denotes the number of columns of (1.1) we see that conditions (1.3) reduce to (1.5) and (1.6) with $a = 1$, $b = 2$, $\alpha = 1$ and $\beta = 0$. It follows that

$$A_{-1}(n) = C_1(n). \tag{3.1}$$

Similarly we can prove that

$$B_{-1}(n) = C_2(n), \tag{3.2}$$

and

$$B_1(n) = C_3(n). \tag{3.3}$$

Now using Burge's theorem 1 ([2], p. 211), we get

$$\begin{aligned}\sum_{n=0}^{\infty} C_1(n)\, q^n &= \sum_{n=0}^{\infty} q^N \lim_{M\to\infty} p(N,M,1,2,1,0,q) \\ &= \sum_{N=0}^{\infty}\sum_{\mu=-\infty}^{\infty} \frac{q^{3\mu^2-2\mu+N}}{(q;q)_{N+3\mu}(q;q)_{N-3\mu}} \\ &\quad -\sum_{N=0}^{\infty}\sum_{\mu=-\infty}^{\infty} \frac{q^{3\mu+4\mu^2+1+N}}{(q;q)_{N+3\mu+1}(q;q)_{N-3\mu+1}}.\end{aligned} \tag{3.4}$$

(3.4) in view of (3.1) and (2.1) with $k = -1$, yields the following analytic identity;

$$\sum_{n=o}^{\infty} \frac{q^{n^2}}{(q;q)_{2n}} = \sum_{N=o}^{\infty} \sum_{\mu=-\infty}^{\infty} \frac{q^{3\mu^2-2\mu+N}}{(q;q)_{N+3\mu}(q;q)_{N-3\mu}}$$

$$(3.5) \qquad - \sum_{N=o}^{\infty} \sum_{\mu=-\infty}^{\infty} \frac{q^{3\mu^2+4\mu+1+N}}{(q;q)_{N+3\mu+1}(q;q)_{N-3\mu+1}}.$$

Similarly we obtain the following analytic identities.

$$\sum_{n=o}^{\infty} \frac{q^{n^2+n}}{(q;q)_{2n}} = \sum_{N=o}^{\infty} \sum_{\mu=-\infty}^{\infty} \frac{q^{3\mu^2+\mu+N}}{(q;q)_{N+3\mu}(q;q)_{N-3\mu}}$$

$$(3.6) \qquad - \sum_{N=o}^{\infty} \sum_{\mu=-\infty}^{\infty} \frac{q^{3\mu^2+\mu+N}}{(q;q)_{N+3\mu+1}(q;q)_{N-3\mu-1}},$$

and

$$\sum_{n=o}^{\infty} \frac{q^{2n^2}}{(q;q)_{2n}} = \sum_{N=o}^{\infty} \sum_{\mu=-\infty}^{\infty} \frac{q^{N}}{(q;q)_{N+3\mu}(q;q)_{N-3\mu}}$$

$$(3.7) \qquad - \sum_{N=o}^{\infty} \sum_{\mu=-\infty}^{\infty} \frac{q^{N}}{(q;q)_{N+3\mu+1}(q;q)_{N-3\mu-1}}.$$

4. Rogers-Ramanujan Type Identities for Burge's Pairs of Partitions

Identities (3.5)–(3.7) in view of (1.12)–(1.14) yield the following Rogers-Ramanujan type identities for Burge's pairs of partitions on MacMohan's lines.

Theorem 4.1. *$C_1(n)$ equals the number of partitions of n into parts* $\not\equiv 0, \pm 2, \pm 6, \pm 8, 10 \pmod{20}$.

Theorem 4.2. *$C_2(n)$ equals the number of partitions of n into parts* $\not\equiv 0, \pm 1, \pm 6, \pm 8, \pm 9, 10 \pmod{20}$.

Theorem 4.3. *$C_3(n)$ equals the number of partitions of n into parts* $\not\equiv 0, \pm 1, \pm 6, \pm 7, \pm 8 \pmod{16}$.

5. Polynominal Identities

Bressoud [1] gave the finite versions of the Rogers-Ramanujan identities. Using Burge's method we give the finite versions of the identities (3.5)–(3.7) in the following forms respectively.

$$\begin{aligned} q^{N^2-N}\begin{bmatrix} N+M+1 \\ 2N \end{bmatrix} &= \sum_{m=-\infty}^{\infty} q^{3\mu^2-2\mu}\begin{bmatrix} N+M+2\mu \\ N+3\mu \end{bmatrix}\begin{bmatrix} N+M-2\mu \\ N-3\mu \end{bmatrix} \\ &- \sum_{\mu=-\infty}^{\infty} q^{3\mu^2+4\mu+1}\begin{bmatrix} N+M+2\mu \\ N+3\mu+1 \end{bmatrix}\begin{bmatrix} N+M-2\mu \\ N-3\mu-1 \end{bmatrix}, \end{aligned} \tag{5.1}$$

$$\begin{aligned} q^{N^2}\begin{bmatrix} N+M \\ 2N \end{bmatrix} &= \sum_{\mu=-\infty}^{\infty} q^{3\mu^2+\mu}\begin{bmatrix} N+M+2\mu \\ N+3\mu \end{bmatrix}\begin{bmatrix} N+M-2\mu \\ N-3\mu \end{bmatrix} \\ &- \sum_{\mu=-\infty}^{\infty} q^{3\mu^2+\mu}\begin{bmatrix} N+M+2\mu+1 \\ N+3\mu+1 \end{bmatrix}\begin{bmatrix} N+M-2\mu-1 \\ N-3\mu-1 \end{bmatrix}, \end{aligned} \tag{5.2}$$

$$\begin{aligned} q^{2N^2-N}\begin{bmatrix} M+1 \\ 2N \end{bmatrix} &= \sum_{\mu=-\infty}^{\infty} \begin{bmatrix} N+M+3\mu \\ N+3\mu \end{bmatrix}\begin{bmatrix} N+M-3\mu \\ N-3\mu \end{bmatrix} \\ &- \sum_{\mu=-\infty}^{\infty} \begin{bmatrix} N+M+3\mu+1 \\ N+3\mu+1 \end{bmatrix}\begin{bmatrix} N+M-3\mu-1 \\ N-3\mu-1 \end{bmatrix}. \end{aligned} \tag{5.3}$$

Proof. (5.2) was proved by Burge in [2]. The proofs of (5.1) and (5.3) run on the same lines. □

Remark. Identity (3.5) can be obtained from (5.1) by first multiplying (5.1) by q^N and then summing over N from $N = 0$ to $N = \infty$ and finally taking lim $M \to \infty$. Similarly, (3.6) can be obtained from (5.2) and (3.7) from (5.3).

6. Conclusion

We have discussed three cases for particular values of a, b, α and β. It would be of interest if for general values of a, b, α and β the Burge's restricted partition pair function can be identified with some generalized Frobenius partition function whose generating function can be found in a form different from Burge's Theorem 1 [2]. That may lead to a generalised analytic identity of which identities (3.5)–(3.7) will be particular cases.

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On q-additive and q-multiplicative Functions

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1. Introduction

Notations.

$\mathbb{N}$ = set of positive integers
$\mathbb{N}_0$ = set of nonnegative integers
$\mathbb{R}$ = set of real numbers
$\mathbb{C}$ = set of complex numbers
$\mathcal{P}$ = set of primes
$e(\alpha) = e^{2\pi i\alpha}$

For some fixed integer $q \geq 2$ and $n \in \mathbb{N}_0$ let $\varepsilon_0(n), \varepsilon_1(n), \dots$ be the sequence of the digits in the q-ary expansion of n, i.e.

$$n = \varepsilon_0(n) + \varepsilon_1(n)q + \dots, \qquad \varepsilon_j(n) \in A_q = \{0, 1, \dots, q-1\}. \tag{1.1}$$

Let G be an arbitrary abelian group, and $\mathcal{A}_q(G)$ be the set of G-valued q-additive functions. A function $\varphi : \mathbb{N}_0 \to G$ belongs to $\mathcal{A}_q(G)$ if $\varphi(0) = 0$ and

$$\varphi(n) = \sum_{j=0}^{\infty} \varphi(\varepsilon_j(n)q^j) \qquad (n \in \mathbb{N}_0). \tag{1.2}$$

For $G = \mathbb{R}$ we write simply $\mathcal{A}_q$ instead of $\mathcal{A}_q(\mathbb{R})$.

A function $g : \mathbb{N}_0 \to \mathbb{C}$ is called q-multiplicative, if $g(0) = 1$ and

$$g(n) = \prod_{j=0}^{\infty} g(\varepsilon_j(n)q^j) \qquad (n \in \mathbb{N}_0) \tag{1.3}$$

Let us denote by $\mathcal{M}_q$ the set of q-multiplicative functions, and $\overline{\mathcal{M}_q} \subseteq \mathcal{M}_q$ be the set of those g, for which additionally $|g(n)| = 1$ $(n \in \mathbb{N}_0)$ holds.

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It is clear that $\varphi(n) = cn$ is q-additive, and $g(n) = z^n$ is q-multiplicative for each $z \in \mathbb{C}$. Furthermore, the sum of digits function, $\alpha(n) := \sum_j \varepsilon_j(n)$ is q-additive as well.

The notion of q-additive functions was introduced by A.O. Gelfond [1]. H. Delange [2] proved that $g \in \mathcal{M}_q$ with the additional condition $|g(n)| \leq 1 (n \in \mathbb{N}_0)$ has a nonzero mean-value m, i.e.

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n<x} g(n) = m, \qquad m \neq 0 \tag{1.4}$$

if and only if

$$\sum_{c\in A_q} g(cq^j) \neq 0 \qquad (j \in \mathbb{N}_0) \tag{1.5}$$

and the series

$$\sum_{j=0}^{\infty} \sum_{c\in A_q} (1 - g(cq^j)) \tag{1.6}$$

is convergent.

If (1.5), (1.6) hold, then

$$m = \prod_{j=0}^{\infty} \frac{1}{q} \left(\sum_{c\in A_q} g(cq^j) \right).$$

Hence he deduced that $f \in \mathcal{A}_q$ has a limit distribution, i.e.

$$\lim_{x\to\infty} \frac{1}{x} \# \{n < x \mid f(n) < y\}$$

converges to a distribution function $F(y)$ for almost all y, if and only if both of the series

$$\sum_{j=0}^{\infty} \sum_{c\in \mathbb{A}_q} f(cq^j), \tag{1.7}$$

$$\sum_{j=0}^{\infty} \sum_{c\in \mathbb{A}_q} f^2(cq^j) \tag{1.8}$$

are convergent.

Delange proved furthermore, that

$$\frac{1}{x}\sum_{n<x} g(n) = \prod_{j=0}^{N_x} \frac{1}{q}\left(\sum_{c\in\mathbb{A}_q} g(aq^j)\right) + o_x(1)$$

where $N_x = [\frac{\log x}{\log q}]$, for all $g \in \mathcal{M}_q$ satisfying $|g(n)| \le 1 (n \in \mathbb{N}_0)$.

Hence one get easily that if $g \in \overline{\mathcal{M}_q}$, then

$$\lim_{x\to\infty} \frac{1}{x}\left|\sum_{n<x} g(n)\right| \qquad (=: \xi)$$

always exists. $\xi \neq 0$ holds if and only if

$$\sum_{j=0}^{\infty}\sum_{c\in A_q} \mathrm{Re}\,(1 - g(cq^j)) < \infty, \tag{1.9}$$

furthermore

$$\xi = \prod_{j=0}^{\infty} \frac{1}{q}\left|\sum_{a\in\mathbb{A}_q} g(aq^j)\right|.$$

2. On the Mean-value of $t(n) = g_1(a_1 n)\dots g_k(a_k n)$ and the Distribution of $l(n)$

Let $a_1 < \cdots < a_k (< q)$ be positive pairwise coprime integers, each of which is coprime to q. Let $g_1, \dots, g_k \in \overline{\mathcal{M}_q}$, $t(n) = g_1(a_1 n)\dots g_k(a_k n)$, $t_j(n) = t(nq^j)$. In a joint paper of Indlekofer and myself [3] we investigated the existence of the mean-value of $t(n)$.

The following assertions are proved.

Let

$$\begin{aligned} M_j(q^N) &= \sum_{n<q^N} t_j(n), \quad m_j(N) := q^{-N} M_j(q^N), \\ \alpha_j &= \liminf_{N\to\infty} |m_j(N)|, \quad \beta_j = \limsup_{N\to\infty} |m_j(N)|. \end{aligned}$$

We proved

Theorem 1. *If $\beta_j > 0$ for some j, then $\alpha_l = \beta_l \to 1$ as $l \to \infty$, furthermore there exist suitable real numbers $\gamma_1, \dots, \gamma_k$ and some $j_0 \ge 0$ such that*

$$q^{j_0}(\gamma_1 a_1 + \cdots + \gamma_k a_k) \equiv 0 \ (mod\ 1), \tag{2.1}$$

and in the notation $h_j(n) := e(-\gamma_j n)\, g_j(n)$,

$$\sum_{j=0}^{\infty} \sum_{c \in A_q} Re\,(1 - h_l(cq^j)) < \infty \tag{2.2}$$

holds for $l = 1, \ldots, k$.

If (2.1), (2.2) *hold, then* $\beta_j > 0$ *for some* j.

Let $f_1, \ldots, f_k \in \mathcal{A}_q$, $a_1, \ldots, a_k$ as above,

$$l(n) = f_1(a_1 n) + \cdots + f_k(a_k n).$$

Theorem 2. *Let* $R \geq 3$. *Then*

$$\max_{j=1,\ldots,k} \max_{n \leq q^{R-3}} |f_j(nq) - nf_j(q)| \leq K \max_{n<q^R} |l(n)|,$$

where K *is a constant which depends on* q^R, $a_1, \ldots, a_k$.

A sequence b_n $(n \in \mathbb{N}_0)$ of real numbers is called a tight sequence, if there is a function $d(x)$ such that

$$c(K) := \limsup_{x \to \infty} \frac{1}{x} \# \{n < x \mid |b_n - d(x)| > K\}$$

tends to zero as $K \to \infty$.

Theorem 3. *Assume that* $l(n)$ *is a tight sequence. Then there exist real numbers* $\gamma_1, \ldots, \gamma_k$ *such that*

$$\gamma_1 a_1 + \cdots + \gamma_k a_k = 0 \tag{2.3}$$

and for $q_t(n) := f_t\,(n) - \gamma_t n (t = 1, \ldots, k)$

$$\sum_{j=0}^{\infty} \sum_{c \in A_q} q_t^2(cq^j) < \infty. \tag{2.4}$$

If (2.3), (2.4) *hold then*

$$\lim \frac{1}{x} \# \{n < x \mid l(n) - d(x) < y\} = F(y) \tag{2.5}$$

exists, F *is a distribution function,*

$$d(x) := \sum_{t=1}^{k} \sum_{cq^j < x} q_t(cq^j). \tag{2.6}$$

Finally, l *has a limit distribution (i.e.* (2.5) *holds with* $d(x) = 0$), *if and only if the right hand side of* (2.6), *and* (2.4) *are convergent.*

Let

$$\begin{aligned}
\pi_u(c_1,\ldots,c_k) &= \frac{1}{q^{u+1}}\#\{n<q^{u+1} \mid \varepsilon_j(a_jn)=c_j, \quad j=1,\ldots,k\},\\
\mu_l(u) &= \frac{1}{q}\sum_{c\in\mathbb{A}} f_l(cq^u),\\
p(u) &= \mu_1(u)+\cdots+\mu_k(u),\\
\sigma_N^2 &= \sum_{u=0}^{N-1}\tau(u).
\end{aligned}$$

Theorem 4. *Let $a_1,\ldots,a_k$ be as earlier, $l(n)=f_1(a_1n)+\cdots+f_k(a_kn)$. Assume that*

$$\max_{l=1,\ldots,k}\max_{c\in\mathbb{A}}\frac{|f_l(cq^M)|}{\sigma_M}\to 0 \quad \text{as} \quad M\to\infty.$$

Then

$$\lim_{x\to\infty}\frac{1}{x}\#\left\{n<x \;\middle|\; \frac{l(n)-p(N_x)}{\sigma_{N_x}}<y\right\}=\Phi(y),$$

where Φ is the Gaussian law, $N_x=[\frac{\log x}{\log q}]$.

3. On a Theorem of H. Daboussi

H. Daboussi [7] proved that if g is a multiplicative function, $|f(n)|\le 1 (n\in\mathbb{N})$, then for each irrational α,

$$\frac{1}{x}\sum_{n\le x} f(n)e(n\alpha)\to 0 (x\to\infty).$$

By using the results of the previous section, in [6] we proved

Theorem 5. *Assume that f is a uniformly summable multiplicative function, i.e.*

$$\limsup_{x\to\infty}\frac{1}{x}\sum_{|f(n)|\ge K}|f(n)|=C(K)\to 0 \text{ as } K\to\infty.$$

Assume that $g\in\overline{\mathcal{M}_q}$ is such a function for which

$$\limsup_{x\to\infty}\frac{1}{x}\left|\sum_{n\le x}f(n)g(n)\right|>0,$$

then there exists a rational number $\frac{r}{D}$, such that $h(n) := g(n)e(-\frac{r}{D}n)$ satisfies

$$\sum_{j=0}^{\infty}\sum_{c\in\mathbb{A}} Re\,(1 - h(cq^j)) < \infty.$$

If the Fourier-Bohr spectrum of f is empty, then

$$\frac{1}{x}\sum_{n\le x} f(n)g(n) \to 0 \qquad (x\to\infty)$$

for each $g \in \overline{\mathcal{M}_q}$.

4. On q-multiplicative Functions Taking a Fixed Value on the Set of Primes

An old conjecture of Gelfond states that for $q = 2$, $\alpha(n) := \sum \varepsilon_j(n)$ (α is the sum of digits function),

$$\frac{1}{\pi(x)}\,\#\,\{p \le x \mid \alpha(p) \equiv 0 \pmod 2\} \to 1/2 \qquad (x\to\infty),$$

where p runs over $\mathcal{P}$. Presently it is not known, whether there do exist infinite subsequences of primes $\{p_j\}$, $\{q_j\}$ such that $\alpha(p_j) =$ even, $\alpha(q_j) = 1$.

Quite plausible is to guess that if $g \in \overline{\mathcal{M}_q}$ takes the value 1 for all but finitely many primes, then $g(nq) = 1$ identically.

In [5] we proved

Theorem 6. *There exists a constant c_1 with the following property. If $g \in \overline{\mathcal{M}_q}$, and $g(p) =$ constant for every large p, then $g^k(nq) = 1$ holds for every $n \in \mathbb{N}_0$, and $k \in [1, c_1]$ is a suitable integer.*

5. Group Valued q-additive Functions on Polynomial Values

In a joint paper of M.V. Subbarao and myself [16] we proved the following Theorems 7, 8.

Theorem 7. *Let $P(n) = A_k n^k + \cdots + A_1 n + A_0$, $A_j \in \mathbb{N}_0$ $(j = 0, 1, \ldots, k)$, $A_k > 0$. Let G be an Abelian group, $f \in \mathcal{A}_q(G)$ with the property that*

$$\lim_{n\to\infty} f(P(n)) = \alpha$$

with some $\alpha \in G$.

Then $f(P(n)) = f(P(0))(n \in \mathbb{N}_0)$ and there exists an integer $v \in \mathbb{N}_0$ such that $f(nq^v) = nf(q^v)$ for $n \in \mathbb{N}_0$.

Let Δ *be the largest common divisor of the values* $\{P(m) - A_0 \mid m \in \mathbb{N}_0\}$. *Let* Δ_1 *be the largest divisor of* Δ *which is coprime to* q. *Let* D_1 *be the largest divisor of* $k!A_k$ *which is coprime to* q.

If $(D_1, \Delta_1) = 1$, *then there exists an integer* $M (\geq 0)$ *such that* $f(bq^u) = 0$ *for every* $b \in \mathbb{A}$ *and* $u \geq M$.

Theorem 8. *Let* q *be odd,* $q \geq 3$. *Let* $f \in \mathcal{A}_q(G)$, *and assume that the set* $\mathcal{E}$ *of the limit points of* $\{f(n^2) \mid n \in \mathbb{N}_0\}$ *is a finite set. Then* $Df(nq^R) = 0$ *with some integers* $D \geq 1$, $R \geq 0$.

6. The Distribution of q-additive Functions on the Set of Primes

In [11] we proved

Theorem 9. *Let* $f \in \mathcal{A}_q$. *Assume that there is a suitable function* $\alpha(x)$ *such that*

$$\lim \frac{1}{\pi(x)} \# \{p \leq x,\ f(p) - \alpha(x) < y\} = F(y), \tag{6.1}$$

where F *is a distribution function. Then* (1.8) *is convergent. Let* $N = N_x = [\frac{\log x}{\log q}]$,

$$\mu(x) := \sum_{j=0}^{N} \frac{1}{q} \sum_{a \in \mathbb{A}} f(aq^j). \tag{6.2}$$

Then $\alpha(x) - \mu(x)$ *tends to a finite limit as* $x \to \infty$. *Especially, if* (6.1) *holds with* $\alpha(x) = 0$, *then* (1.7) *is convergent as well.*

On the other hand, the convergence of (1.7), (1.8) *guaranties the fulfilment of* (6.1) *with* $\alpha(x) \equiv 0$.

7. Mean Values of q-additive Functions on the Set of Primes

Let $g \in \overline{\mathcal{M}_q}$,

$$P(x) = \sum_{p \leq x} g(p), \quad S(x) = \sum_{\substack{l < x \\ (l,q)=1}} g(l)e(\alpha l).$$

We are interested in to give necessary and sufficient conditions for g to satisfy

$$\lim_{x \to \infty} \frac{P(x)}{\pi(x)} = 0. \tag{7.1}$$

Conjecture 1. (7.1) holds if and only if

$$x^{-1}S(x \mid r) \to 0 \tag{7.2}$$

for every $r \in \mathbb{Q}$.

The necessity of (7.2) is quite easy, since if it does not hold then

$$\sum_{j=0}^{\infty}\sum_{c\in\mathbb{A}} \operatorname{Re}\,(1-g(cq^j)e(cq^jr)) < \infty.$$

The difficulty is in the sufficiency.

Let

$$\begin{aligned} T_{l_1,l_2}^{(M)} &= T_{l_1,l_2} \\ &= \#\{p_1,p_2\in\mathcal{P},\ p_2-p_1=l_2-l_1,\ p_1\equiv l_1(\operatorname{mod} q^M),\ p\le x\} \\ H(d):&=\prod_{\substack{p\mid d\\ p\nmid 2q}}\left(1+\frac{1}{p-2}\right). \end{aligned}$$

Conjecture 2. There exists a constant $0<\delta<1/2$, such that with $M=[\delta N]$, $N == [\frac{\log x}{\log q}]$,

$$\sum_{\substack{l_1,l_2<q^M\\ (l_1l_2,q)=1\\ l_1\ne l_2}}\left|T_{l_1,l_2}^{(M)}-\frac{x}{\varphi(q^M)\log^2 x}H(l_2-l_1)\right| < \frac{\varepsilon(x)xq^M}{\log^2 x}$$

with a suitable function $\varepsilon(x)\to 0 \quad (x\to\infty)$.

We proved that Conjecture 2 implies the fulfilment of Conjecture 1.

Let $Y(x)$ be a monotone increasing function tending to infinity as $x\to\infty$, such that $\frac{\log Y(x)}{\log x}\to 0$. Let $\mathcal{N}_x := \{n\le x,\ p(n)>Y(x)\}$, where $p(n)$ is the smallest prime factor of n. Let $N(x) =$ card $(\mathcal{N}_x)$. Let L be the strongly multiplicative function which for primes p is defined as follows:

$$L(p)=\begin{cases}\frac{1}{p-2} & \text{if } p>2\ p\nmid q\\ 0 & \text{otherwise}\end{cases}$$

In [11] we proved

Theorem 10. *Let* $g\in\overline{\mathcal{M}_q}$,

$$U(x):=\sum_{n\in\mathcal{N}_x} g(n).$$

Then

$$\left|\frac{U(x)}{N(x)}\right|^2 \le \sum_{d<D} \frac{L(d)}{d} \sum_{a=0}^{d-1} \left|q^{-M} S\left(q^M \,\middle|\, \frac{a}{d}\right)\right|^2 + \frac{c_1}{D} + o_x(1),$$

where $M = [\frac{1}{4}\frac{\log x}{\log q}]$, c_1 *is a positive constant which depends only on* q, $o_x(1)$ *does depend on* $Y(x)$, *and* D *is an arbitrary real number.*

As a consequence we have that (7.2) implies that $U(x) = o_x(1)N(x)$.

9. The Distribution of q-ary Digits on Some Subsets of Integers

Let $\mathcal{B}$ be an infinite sequence of nonnegative integers, $B(x) = \#\{b \le x,\ b \in B\}$. For $0 \le l_1 < l_2 < \cdots < l_h (\le N), b_1, \dots, b_h \in \mathbb{A}$ let $\mathbb{A}_{\mathcal{B}}\left(x \,\middle|\, \begin{matrix} l \\ b \end{matrix}\right)$ be the size of the set of integers $n \le x$, for which $n \in \mathcal{B}$, and $\varepsilon_{l_j}(n) = b_j (j = 1, \dots, h)$ simultaneously hold.

Conjecture 3. *For each choice of* $(1 \le) l_1 < \cdots < l_k$ $(\le N)$ *and* $b_1, \dots, b_h \in \mathbb{A}$,

$$\text{(9.1)} \qquad \sup_{1 \le h \le \frac{1}{3}N} \ \sup_{\substack{l_1,\dots,l_h \\ b_1,\dots,b_h}} \left| \frac{q^h A_{\mathcal{P}}\left(q^N \,\middle|\, \begin{matrix} l \\ b \end{matrix}\right)}{\pi(q^N)} - 1 \right| \to 0 \qquad (N \to \infty).$$

Here $\mathcal{P}$ is the set of primes.

Remarks.

1. The inequality (9.1) in the much weaker form, $\sup_{1 \le h \le \log\log N}$ instead of $\sup_{1 \le h \le 1/3N}$ was proved in [14].
2. (9.1) for the extremal positions of $l_1, \dots, l_h$, i.e. for $l_h \le N/3$, and for $l_1 \ge N - N/3$ are known to be true. The first assertion is proved by R. Heath-Brown, the second follows from the prime-number theorem for short intervals.

In a joint paper of N.L. Bassily and myself [9] we proved

Theorem 11. *Let* $f \in \mathcal{A}_q$, *such that* $f(bq^j)$ *is bounded if* hq^j *runs over* $\mathcal{R}$. *Let* $P(x)$ *be a polynomial with integer coefficients, the leading coefficient of which is positive. Let* $r = \deg P(x)$, $N = [\frac{\log x}{\log q}]$,

$$m_k := \frac{1}{q} \sum_{b \in \mathbb{A}} f(bq^k), \quad \sigma_k^2 := \frac{1}{q} \sum_{b \in E} f^2(bq^k) - m_k^2$$

$$M(x) := \sum_{k=0}^{N} m_k, \qquad D^2(x) := \sum_{k=0}^{N} \sigma_k^2.$$

Assume that $\frac{D(x)}{(\log x)^{1/3}} \to \infty$.

Then

$$\frac{1}{x}\ \#\ \left\{n < x \,\middle|\, \frac{f(P(n)) - M(x^r)}{D(x^r)} < y\right\} \to \Phi(y),$$

$$\frac{1}{\pi(x)}\ \#\ \left\{p < x \,\middle|\, \frac{f(gP(p)) - M(x^r)}{D(x^r)} < y\right\} \to \Phi(y)$$

where Φ is the normal distribution function.

The proof is based upon some theorems of I.M. Vinogradov and L.K. Hua for trigonometric sums, by which we derived

Lemma 1. *Let $\lambda > 0$ be an arbitrary constant. Let h be fixed, $b_1, \dots, b_h \in \mathbb{A}$,*

$$N^{1/3} \le l_1 < l_2 < \cdots < l_h \le rN - N^{1/3}. \tag{9.2}$$

Let

$$\begin{aligned}\textstyle\sum_1 : &= \#\{n \le x \mid a_{l_j}(P(n)) = b_j, \quad j = 1, \dots, h\},\\ \textstyle\sum_2 : &= \#\{p \le x \mid a_{l_j}(P(p)) = b_j, \quad j = 1, \dots, h\}.\end{aligned}$$

Then

$$\sum\nolimits_1 = \frac{x}{q^h} + O\left(x(\log x)^{-\lambda}\right), \quad \sum\nolimits_2 = \frac{x}{g^h} + O(x(\log x)^{-\lambda})$$

uniformly in l_j subject to (9.2). The implicit constants in the error terms may depend on the polynomial P, on h, and on λ.

Let

$$\begin{aligned}a_k(x) &= \frac{1}{x}\sum_{n\le x}\left(\frac{f(P(n)) - M(x^r)}{D(x^r)}\right)^k,\\ b_k(x) &= \frac{1}{\pi(x)}\sum_{p\le x}\left(\frac{f(P(p)) - M(x^r)}{D(x^r)}\right)^k,\\ c_k(x) &= \frac{1}{x^r}\sum_{n\le x^r}\left(\frac{f(n) - M(x^r)}{D(x^r)}\right)^k.\end{aligned}$$

From Lemma 1 almost immediately follows that $a_k(x) - c_k(x) \to 0$, $b_k(x) - c_k(x) \to 0$. By standard methods of probability theory for sums of independent random variables, one can get that for every integer $k \ge 0$,

$$\lim c_k(x) = \int x^k d\phi,$$

whence by the Frechet-Shohat theorem, Theorem 11 follows.

10. q-additive Functions Over the Ring of Gaussian Integers

Let $\mathbb{Z}[i]$ be the set of Gaussian integers, $\Theta = a + bi$, $|\Theta|^2 \geq 2$, $\mathcal{A} = \{a_0 = 0, a_1, \dots, a_{t-1}\}(t = |\Theta|^2)$ be a complete residue system mod Θ. Then, for each $\alpha \in \mathbb{Z}[i]$ there is a unique b_0, and $\alpha_1 \in \mathbb{Z}[i]$, such that $\alpha = b_0 + \Theta\alpha_1$. Let $J : \mathbb{Z}[i] \to \mathbb{Z}[i]$ be defined by $J(\alpha) = \alpha_1$.

Iterating J, we define the orbit

$$\alpha(=\alpha_0), \quad \alpha_1 = J(\alpha_0), \quad \alpha_2 = J(\alpha_1), \dots . \tag{10.1}$$

Let

$$L := \frac{1}{|\Theta| - 1} \max_{a \in \mathcal{A}} |a|.$$

It is easy to show that

(a) if $|\alpha| > L$, then $|\alpha_1| < |\alpha|$,

(b) if $|\alpha| \leq L$, then $|\alpha_1| \leq L$.

Hence it follows that the orbit (10.1) is ultimately periodic for every $\alpha \in \mathbb{Z}[i]$.

Let $\mathcal{P}^*$ be the set of periodic numbers. $\pi \in \mathbb{Z}[i]$ is said to be periodic, if $J^l(\pi) = \pi$ for some $l \in \mathbb{N}$.

Repeating the expansion $\alpha = b_0 + \Theta\alpha_1$, we obtain that

$$\alpha = b_0 + b_1\Theta + \dots + b_{k-1}\Theta^{k-1} + \Theta^k\alpha_k \qquad (k = 1, 2, \dots). \tag{10.2}$$

Let k be the smallest nonnegative integer for which $\alpha_k \in \mathcal{P}^*$. The expansion (1.2) with such a k is called the "correct expansion of α".

Assume that $\Theta, \mathcal{A}$ are fixed.

A function $f : \mathbb{Z}[i] \to \mathbb{R}$ is additive (with respect to the expansion generated by Θ and $\mathcal{A}$), if

(a) $f(\pi\Theta^k) = 0$ for $\pi \in \mathcal{P}^*$ and $k = 0, 1, 2, \dots,$
and

(b) for every $\alpha \in \mathbb{Z}[i]$,

$$f(\alpha) = f(b_0) + f(b_1\Theta) + \dots + f(b_{k-1}\Theta^{k-1}),$$

where $\alpha = b_0 + b_1\Theta + \dots + b_{k-1}\Theta^{k-1} + \Theta^k\pi$ is the correct expansion of α.

Let $\mathcal{E}_\Theta$ be the class of additive functions in the above sense.

A function $g : \mathbb{Z}[i] \to \mathbb{C}$ is multiplicative (with respect to the expansion generated by Θ and $\mathcal{A}$), if

(a) $g(\pi\Theta^k) = 1$ for $\pi \in \mathcal{P}^*$ and $k = 0, 1, 2, \dots,$
and

(b) for every $\alpha \in \mathbb{Z}[i]$,

$$g(\alpha) = \prod_{j=0}^{k-1} g(b_j \Theta^j).$$

Let $\mathcal{M}_\Theta$ be the class of multiplicative functions in the above sense, and $\overline{\mathcal{M}}_\Theta (\subseteq \mathcal{M}_\Theta)$ be the set of those g for which additionally $|g(\alpha)| = 1 \quad (\alpha \in \mathbb{Z}[i])$ holds.

In a joint paper of P. Liardet and my [15] we showed that the analogons of the theorems of H. Delange remain valid.

For an interval $I \subseteq [-1/2, 1/2)$ let C_I denote the annulus $\{z \mid z \in \mathbb{C}, \frac{1}{|\Theta|} < |z| < 1, \frac{\arg z}{2\pi} \in I\}$. Let $g \in \overline{\mathcal{M}}_\Theta$, and

$$S_I(x \mid g) := \sum_{\alpha \in xC_I} g(\alpha),$$

where x is a positive growing parameter, $xC_I = \{xz \mid z \in C_I\}$.

It is wellknown that $S_I(x \mid 1)$ = number of Gaussian integers in xC_I is $\pi |I| x^2 (1 - 1/t) + O(x)$ as $x \to \infty$, uniformly in I.

Let

$$N_x = \frac{\log x}{\log |\Theta|},$$

$$\Delta_j := \sum_{b \in \mathcal{A}} g(b\Theta^j). \tag{10.3}$$

Theorem 12. *Let* $g \in \overline{\mathcal{M}}_\Theta$.

1) *If the series*

$$\sum_{j=0}^{\infty} \sum_{c \in \mathcal{A}} Re\,(1 - g(c\Theta^j)) \tag{10.4}$$

is divergent, then

$$\frac{S_I(x \mid g)}{S_I(x \mid 1)} \to 0 \quad \text{as} \quad x \to \infty,$$

uniformly on the intervals I *the length of which is bounded below by a positive constant.*

Consequently,

$$\frac{1}{\pi x^2 |I|} \sum_{\substack{|\alpha| \le x \\ \frac{\arg \alpha}{2\pi} \in I}} g(\alpha) \to 0.$$

2) *If* (10.4) *is convergent, then*

$$\lim_{x\to\infty}\left|\frac{S_I(x\mid g)}{S_I(x\mid 1)}\right| = \prod_{j=0}^{\infty}\frac{1}{t}|\Delta_j|,$$

the right hand side is nonzero, if $\Delta_j \neq 0$ *holds for every* j.

3) *The nonzero limit*

$$\lim_{x\to\infty}\frac{S_I(x\mid g)}{S_I(x\mid 1)} \qquad (= m)$$

exists, if and only if

$$\sum_{j=0}^{\infty}\sum_{c\in\mathcal{A}}(1-g(c\Theta^j))$$

is convergent, and $\Delta_j \neq 0 \quad (j = 0, 1, 2, \ldots)$.

As a consequence we have

Theorem 13. *Let* $f \in \mathcal{A}_\Theta$ *and assume that it has a limit distribution, i.e. that*

$$\lim_{x\to\infty}\frac{1}{\pi x^2}\,\#\,\{|\alpha| \le x \mid f(\alpha) < y\} = F(y)$$

exists, where F *is a distribution function. Then both of the series*

$$\sum_{j=0}^{\infty}\sum_{c\in\mathcal{A}} f(c\Theta^j) \tag{10.5}$$

$$\sum_{j=0}^{\infty}\sum_{c\in\mathcal{A}} f^2(c\Theta^j) \tag{10.6}$$

are convergent.

If (10.5), (10.6) *are convergent, then for each* $I \subseteq [-1/2, 1/2)$,

$$\lim_{x\to\infty}\frac{1}{\pi x^2|I|}\,\#\left\{|\alpha| \le x \mid f(\alpha) < y,\ \frac{\arg\alpha}{2\pi} \in I\right\} = F(y).$$

The characteristic function φ *of* F *can be given by*

$$\varphi(\tau) = \prod_{j=0}^{\infty}\left\{\frac{1}{t}\sum_{c\in\mathcal{A}} e^{i\tau f(c\Theta^j)}\right\}.$$

Another corollary of Theorem 13 is

Theorem 14. *Let* $f \in \mathcal{A}_\Theta$, $f(c\Theta^j) = O(1)$ *as* $j \to \infty$, $c \in \mathcal{A}$, $m_j = \frac{1}{t}\sum_{c\in\mathcal{A}} f(c\Theta^j), \sigma_j^2 = \frac{1}{t}\sum_{c\in\mathcal{A}}(f(c\Theta^j) - m_j)^2$, $E_N := \sum_{j=0}^{N} m_j$, $T_N^2 := \sum_{j=0}^{N}\sigma_j^2$. *Assume that* $T_N \to \infty$. *Let* $I \subseteq [-1/2, 1/2)$. *Then*

$$\frac{1}{\pi x^2 |I|}\,\#\left\{|\alpha| < x,\ \frac{\arg\alpha}{2\pi} \in I,\ \frac{f(\alpha) - E_{N_x}}{T_{N_x}} < y\right\} = (1 + \sigma_x(1))\Phi(y).$$

Φ *is the Gaussian law.*

Theorem 15. *Let* $f \in \mathcal{E}_\Theta$, $f(c\Theta^j) = f(c)$ *=rational integer for* $c \in \mathcal{A}$, *and assume that the greatest common divisor of* $f(c)$ $(c \in \mathcal{A})$ *is* 1. *Let* $m = \frac{1}{t}\sum f(c), \sigma^2 = \frac{1}{t}\sum(f(c) - m)^2$. *Let* $N_I(x \mid k)$ *be the number of the Gaussian integers* α *satisfying* $f(\alpha) = k$ *in the sector* $|\alpha| \le x$, $\frac{\arg\alpha}{2\pi} \in I$. *Then*

$$N_I(x \mid k) = \pi |I| \cdot x^2 \left\{\frac{1}{\sqrt{N_x}}\varphi\left(\frac{k - mN_x}{\sigma\sqrt{N_x}}\right) + O\left(\frac{(\log N_x)^3}{N_x}\right)\right\}$$

where $N_x = [\frac{\log x}{\log|\Theta|}]$, *and* $\varphi(y) = \frac{1}{\sqrt{2\pi}}e^{-y^2/2}$.

A nice theorem is due to B. Gittenberg and J.M. Thuswaldner [17], which we quote as

Theorem 16. *Assume that* $\mathcal{A} = \{0, 1, \ldots, t-1\}$, $t = |\Theta|^2$, *and that* $\mathcal{P}^* = \{0\}$. *Let* $f \in \mathcal{E}_\Theta$, $f(c\Theta^j) = 0(1)$ *for* $j \in \mathbb{N}$, $c \in \mathcal{A}$. *Assume that* $T_N/(\log N)^{1/3} \to \infty$ $(N \to \infty)$. *Let* $P(z) = p_r z^r + \cdots + p_1 z + p_0$ *be a polynomial with integer coefficients. Then, as* $N \to \infty$,

$$\frac{1}{\pi N}\,\#\left\{|z|^2 < N \,\middle|\, \frac{f(P(z)) - E_{rN}}{T_{N_r}} < y\right\} \to \Phi(y),$$

where Φ is the normal distribution function.

11. Generalized Number Systems and the Analogons of q-additive, q-multiplicative Functions Defined by the Expansion

Let M be a $k \times k$ type matrix with integer entries, the k distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ of which are located outside of the closed unit disc, i.e. $|\lambda_j| > 1$ $(j = 1, \ldots, k)$. Thus M defines a map, $\mathbb{Z}_k \to \mathbb{Z}_k$. Let $\mathcal{L} = M\mathbb{Z}_k$. Then $\mathcal{L}$ is a subgroup in $\mathbb{Z}_k$, $O(\mathbb{Z}_k \mid L) == |\lambda_1 \ldots \lambda_k| = |\det M|$.

Let us choose an arbitrary coset representative $\mathcal{A} = \{\underline{a}_0 = \underline{0}, \underline{a}_1, \ldots, \underline{a}_{t-1}\}$, $t = |\det M|$. Then, for each $\underline{n} \in \mathbb{Z}_k$ there is a unique $\underline{n}_1 \in \mathbb{Z}_k$ and a unique $\underline{b}_0 \in \mathcal{A}$ such that $\underline{n} = \underline{b}_0 + M\underline{n}_1$. We define the function $J : \mathbb{Z}_k \to \mathbb{Z}_k$, by $J(n) = n_1$. Let J^l be the l'th iterate of J. Some element $\pi \in \mathbb{Z}_k$ is called periodic, if $J^l(\pi) = \pi$ holds for some $l \in \mathbb{N}$. Let $\mathcal{P}^*$ be the set of periodic element.

One can prove easily that $\mathcal{P}^*$ is a finite set, it always contains the zero, i.e. $\underline{0} \in \mathcal{P}^*$. Furthermore, starting from some $\underline{n} \in \mathbb{Z}_k$, the sequence $n,\ J(n),\ J^2(n), \dots$ is ultimately periodic.

Let R be the smallest integer for which $J^R(\underline{n}) \in \mathcal{P}^*$, and let $J^R(n) = \pi(n)$. Then n has a representation of form:

$$\underline{n} = b_0(\underline{n}) + Mb_1(\underline{n}) + \cdots + M^{R-1}b_{r-1}(\underline{n}) + M^R\pi(\underline{n}). \tag{11.1}$$

We say that (11.1) is the "correct representation" of $\underline{n}$.

Let M and $\mathcal{A}$ be fixed. The set of $(\mathcal{A}, M)$ additive functions, multiplicative functions are denoted by $\mathcal{E}_{\mathcal{A},\mathcal{M}}$ and $\mathcal{M}_{\mathcal{A},M}$, respectively.

We say that $f\ :\ \mathbb{Z}_k \to \mathbb{R}$ belongs to $\mathcal{E}_{\mathcal{A},\mathcal{M}}$, if $f(M^r\pi) = 0$ for every $\pi \in \mathcal{P}^*$, $r = 0, 1, 2, \dots$, and for ever $\underline{n} \in \mathbb{Z}_k$:

$$f(\underline{n}) = \sum_{j=0}^{R-1} f(M^j b_j(n)).$$

A function $g\ :\ \mathbb{Z}_k \to \mathbb{C}$ is in $\mathcal{M}_{\mathcal{A},M}$, if $g(M^r\pi) = 1$ for $\pi \in \mathcal{P}^*$, $r = 0, 1, 2, \dots$, and for every $n \in \mathbb{Z}_k$:

$$g(n) = \prod_{j=0}^{R-1} g(M^j b_j(n)).$$

There are a lot of interesting questions which can be investigated, the following is a typical one.

Let S be an arbitrary domain in $\mathbb{R}_k$. Let

$$S_x = \{(xx_1, \dots, xx_n) \mid (x_1, \dots, x_n) \in S\}.$$

Let $g \in \mathcal{M}_{\mathcal{A},M}$ with $|g(\underline{n})| = 1 \quad (n \in \mathbb{Z}_k)$,

$$B_x(\underline{g}) = \sum_{\underline{n} \in S_k} g(\underline{n}), \quad B_x(1) = \#\{\underline{n} \in S_x\}.$$

1) Give necessary and sufficient conditions for g to satisfy

$$(\alpha) \qquad \frac{B_x(g)}{B_x(1)} \to 0 \qquad (x \to \infty)$$

$$(\beta) \qquad \lim \left|\frac{B_x(g)}{B_x(1)}\right| = \text{exists}, \neq 0,$$

$$(\gamma) \qquad \lim \frac{B_x(g)}{B_x(1)} = \text{exists}, \neq 0.$$

2) Give necessary and sufficient conditions for $f \in \mathcal{E}_{\mathcal{A},M}$ to possess a limit distribution:

$$\lim_{x\to\infty} \frac{1}{B_x(1)} \# \{\underline{n} \in S_x \mid f(\underline{n}) < y\} = F(y),$$

F is a distribution function.

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Antimagic Labeling of Complete m-ary Trees

P.D. Chawathe and Vijaya Krishna

A function f is called an *antimagic labeling* of a graph G with q edges if f is an injection from the edges of G to the set $\{1, 2, 3, \dots, q\}$ such that when each vertex v is assigned the sum of all the labels of edges incident with v, the resulting vertex labels are all distinct. It is shown that all complete m-ary trees have antimagic labelings.

1. Introduction

Let G be any graph with vertex set $V(G)$ and edge set $E(G)$. For any vertex $x \in V(G)$ and any edge $e \in E(G)$, we write $e \sim x$, if they are incident. Let $\mathbb{N}$ denote the set of natural numbers. For any function $f : E(G) \mapsto \mathbb{N}$, the induced function $\overline{f} : V(G) \mapsto \mathbb{N}$ is defined as follows: for each vertex $v \in V(G)$, $\overline{f}(v) = \sum_{e \sim v} f(e)$.

Hartsfield and Ringel [1] introduced the concept of antimagic labeling. Let G be a graph with q edges. A bijection f from $E(G)$ to $\{1, 2, 3, \dots, q\}$ is called an *antimagic labeling* of G if $\overline{f}$ is one-one. They have made the following conjecture: Every tree has an antimagic labeling. Towards this we prove that complete m-ary trees have antimagic labelings.

2. Main Result

A complete m-ary tree ($m \geq 2$) of height h is a tree of order $1+m+m^2+\cdots+m^h$. It has a unique vertex of degree m called the root. It has m^h pendant vertices, each pendent vertex being at distance h from the root and the remaining vertices have degree $m + 1$. We prove the following result.

Theorem. *Every complete m-ary tree has an antimagic labeling.*

Proof. We denote the root of the m-ary tree by 0; vertices which are adjacent to 0 by $1, 2, \dots, m$; vertices which are adjacent to 1 and at distance two from 0 by $m + 1, m + 2, \dots, 2m$; vertices which are adjacent to 2 and at distance two from 0 by $2m + 1, 2m + 2, \dots, 3m; \dots$; vertices which are adjacent to m and at distance two from 0 by $m^2 + 1, m^2 + 2, \dots, m^2 + m$. We then list the

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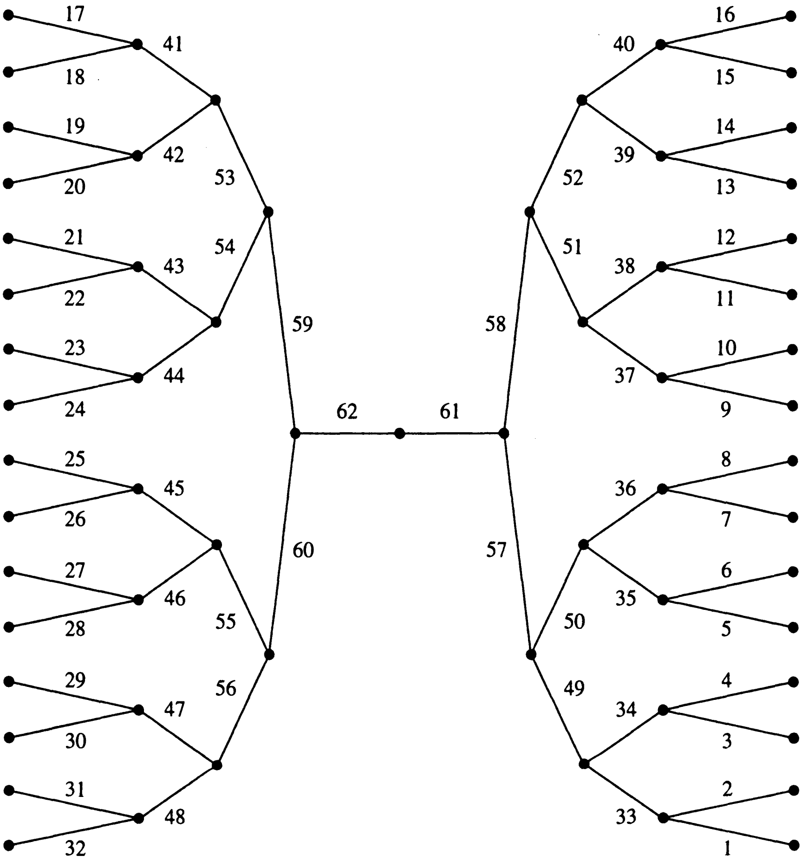

Figure 1

vertices which are at distance three from 0 by first listing the vertices which are adjacent to $m+1$, then listing the vertices which are adjacent to $m+2$ and so on. Thus the vertex set of the tree is $V = \{0, 1, 2, \ldots, p-1\}$, where $p = 1 + m + m^2 + \cdots + m^h$.

If ij is an edge and distance of i from 0 is less than distance of j from 0, then we say that i is a "parent" of j and j is a "child" of i. In this case we denote i by parent (j). Thus each vertex $i \leq m + m^2 + \cdots + m^{h-1}$ has m children: $mi+1, mi+2, \ldots, mi+m$ and each vertex $i \neq 0$ has $\lfloor \frac{i-1}{m} \rfloor$ as its parent. Any edge of the tree can be represented as parent $(i)i$, where $i \in V\backslash\{0\}$. We now label the edges by

$$f(\text{parent } (i)i) = p - i \text{ for all } i \in V\backslash\{0\}.$$

Then

$$\begin{aligned} \bar{f}(0) &= (p-1) + (p-2) + \cdots + (p-m) \\ &= mp - (1 + 2 + \cdots + m), \end{aligned}$$

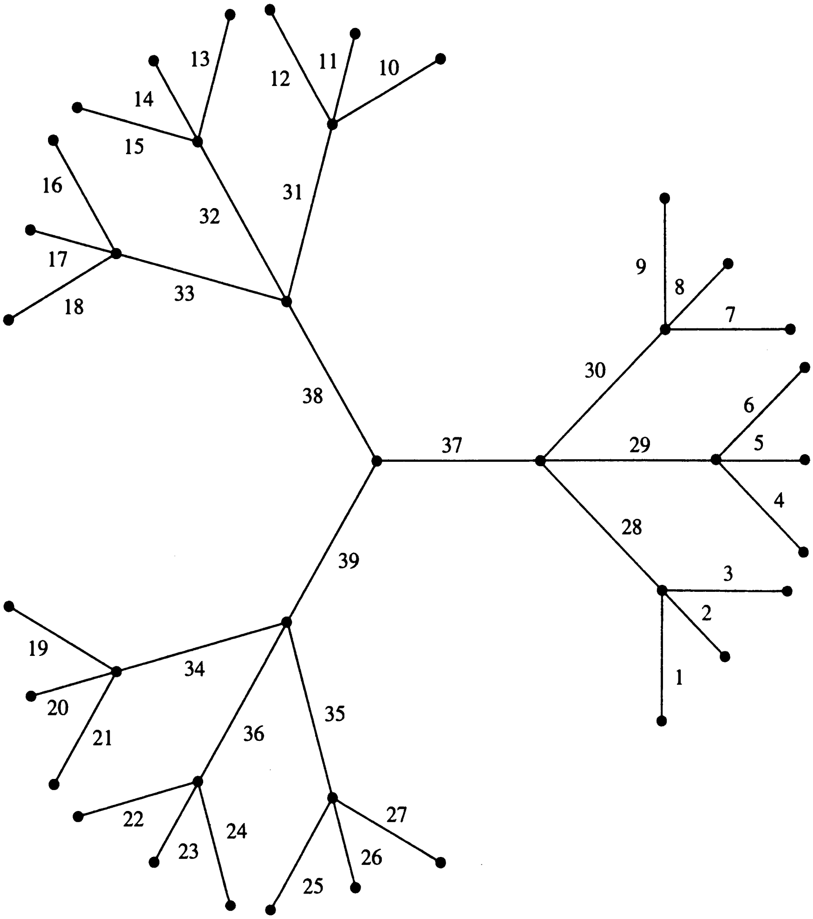

Figure 2

$$\begin{aligned} \bar{f}(i) &= (p-i)+(p-mi-1)+(p-mi-2)+\ldots(p-mi-m)\\ &= (m+1)p-(m^2+1)i-(1+2+\cdots+m)\\ &\quad \text{for } 1\le i\le m+m^2+\cdots+m^{h-1},\\ \bar{f}(i) &= p-i \text{ for } 1+m+m^2+\cdots+m^{h-1}\le i\le p-1. \end{aligned}$$

Clearly, $\bar{f}(1), \bar{f}(2), \ldots, \bar{f}(p-1)$ are all mutually distinct. Also $\bar{f}(0)$ is distinct from $\bar{f}(i)$ for $1+m+m^2+\cdots+m^{h-1}\le i\le p-1$. Suppose $\bar{f}(0)=\bar{f}(i)$ for some i, $1\le i\le m+m^2+\cdots+m^{h-1}$. Then $p=(m^2+1)i$. Let $h=4k+r$ with $0\le r\le 3$. It is easy to verify that $p=(m^2+1)i$ iff $r=3$ and

$$i=1+m+m^4+m^5+\cdots+m^{4k}+m^{4k+1}. \tag{1}$$

So the given labeling is an anti-magic labeling, if $h \not\equiv 3\ (mod\ 4)$. If $h = 3\ (mod\ 4)$, the vertex labels are all mutually distinct except $\bar{f}(0) = \bar{f}(i)$, where i is given by equation (1). Let j = parent (i). Choose a child s of the vertex j such that $| s - i | = 1$. Then labels of the edges js and ji differ by one. Now interchange the labels of edges js and ji. This does not change the label of the vertex j; the labels of vertices s and i change by 1. Since the difference between old labels of vertices s and i i.e. the difference between $\bar{f}(s)$ and $\bar{f}(i)$ is at least $m^2 + 1 \geq 5$, the new labels of the vertices s and i are distinct. This new labeling is clearly antimagic. □

We illustrate the labeling by the following examples.

Example 1. $m = 2, h = 5, q = 62$.

Since $h \not\equiv 3\ (mod\ 4)$, no edge label is to be changed. See Figure 1.

Example 2. $m = 3, h = 3, q = 39$.

Edge labels 35 and 36 are interchanged. See Figure 2.

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Some Recent Advances on Symmetric, Quasi-Symmetric and Quasi-Multiple Designs

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Introduction

Since the advent of design theory that began with constructive results of R.C. Bose and M. Hall, symmetric designs have occupied a very special status. This is due to several reasons, two most important of which are the structural symmetry and the difficulty in constructions of these designs (compared to other classes of designs). The initial part of this exposition will concentrate on new results on symmetric designs. Quasi-symmetric designs are closely related to symmetric designs. These are designs that have (at the most) two block intersection numbers. One can associate a block graph with a quasi-symmetric design and in many cases of interest, this also turns out to be a graph with special properties, and is called a *strongly regular graph.* One trivial way of constructing quasi-symmetric designs is to take multiple copies of a symmetric design. Since a symmetric design has exactly one block intersection number, the resulting design will have two block intersection numbers. Such quasi-symmetric designs are called *improper.* Only one example of a *proper* quasi-multiple quasi-symmetric design seems to be known so far. The question of the existence of a quasi-multiple of a symmetric design is of some importance particularly when the corresponding symmetric design is known not to exist. In that case, obtaining a quasi-multiple with least multiplicity has attracted attention of combinatorialists in the last thirty years.

This paper is organized in four (small) sections. In the first section, we introduce symmetric designs and discuss some recent results of Ionin on new parameter sets and a result of the author concerning non-isomorphic designs with particular parameter sets. In the second section, we discuss quasi-multiples of symmetric designs and residual designs. In the third section, we discuss quasi-symmetric designs. In the fourth and the last section, we give connection of these structures and in particular, discuss a recent proof of the Jungnickel-Tonchev conjecture by the author.

The standard references are the books by Beth, Jungnickel and Lenz and by M. Hall. Some other recent references are also given.

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Symmetric Designs

By an incidence structure, we mean a triple, consisting of the set of points, the set of blocks (a block is nothing but a distinguished subset of the point-set) and the incidence (a point belongs or does not belong to a block). Repetition of blocks may sometimes be allowed in later sections but will not occur in this section, where we discuss symmetric designs. An incidence structure is called a *design* (balanced incomplete block design or a 2-design) if every block has a constant size k, which is strictly less than v, the number of points, and any two points occur together (are commonly contained in) the same number λ of blocks. If v is the number of points, b the number of blocks and r the number of blocks containing a given point, then such a configuration is called a (v, b, r, k, λ)-design or sometimes also called a (v, k, λ)-design. The following basic inequality is satisfied by any design:

Fisher's inequality We always have $v \leq b$, i.e., the number of points is less than or equal to the number of blocks. Further if D is a design with $v = b$, the incidence structure in which blocks are points and points are blocks is also a design with the same parameter set (v, k, λ). A design with $v = b$ is called a *symmetric design.*

Fisher's inequality asserts that the dual of a symmetric design is also a symmetric design. In fact, we have:

Theorem. *The following are equivalent for any design D.*

1. *D is a symmetric design.*
2. $r = k$.
3. *Any two blocks of D intersect in λ points.*
4. *The dual of D is also a design.*

For the reasons of structural symmetry and strong connections with group theory, symmetric designs are objects of considerable interest. Many constructions of symmetric designs are known. There are two main classes of designs obtained in a "classical way": Designs coming from projective geometries and designs coming from Hadamard matrices. The only known general necessary conditions for the existence of symmetric designs are those due to Bruck Ryser and Chowla (see [1, 4]). These were sometimes believed to be sufficient, but then a computer result of Lam and others [10] provided a counterexample: a putative projective plane of order ten satisfies the Bruck-Ryser-Chowla conditions *but it does not exist.*

A large number of constructions of symmetric designs are based on "difference sets". A difference set in a group is a *subset* of a group satisfying certain stipulated conditions. The existence of a difference set is equivalent to the existence of a symmetric design (with corresponding parameters) which admits a regular automorphism group. Continuing the discussion in the previous paragraph, two important families of difference sets are the Singer difference sets (corresponding to a regular group action on the hyperplanes and points of a projective geometry) and the Hadamard difference sets. In spite of a lot of knowledge regarding constructions of symmetric designs, the following conjecture seems to be a difficult unsolved problem in design theory.

M. Hall's Conjecture. (see [22]) For a given $\lambda \geq 2$ there are only finitely many symmetric designs with parameter set (v, k, λ).

In fact, for a fixed $\lambda \geq 4$, based on the present knowledge one can make a stronger conjecture that $k \leq \lambda^2 + \lambda$. In a recent seminal paper on this difficult problem of constructions of symmetric designs, Ionin [6, 7] has obtained an elegant general method that yields about ten different families of symmetric designs not known until three years ago. Improving very strongly on the methods of Dinesh Rajkunlia where the use of weighing matrices and designs was first made, Ionin's constructions have far reaching consequences. Some simplifications of Ionin constructions are published in the last couple of years and the author also expects to write out some simplification of Ionin constructions. We give below one of the theorems of Ionin [6].

Theorem 1.1. *There exist symmetric designs D with parameters:*

$$v = 1 + 2(q+1)\frac{(q+1)^{2m} - 1}{q+2}, \; k = (q+1)^{2m}, \; \lambda = \frac{(q+1)^{2m-1}(q+2)}{2}$$

where $q = 2^p - 1$ *is a Mersene prime and* m *is a positive integer.*

For a given parameter set v, k, λ how many non-isomorphic symmetric designs can be constructed? A general answer to this question is not known, though it has been shown that asymptotically one can obtain an exponential lower bound on the number of non-isomorphic designs belonging to a particular family of symmetric designs. It is known, for example, that we have an exponential lower bound on non-isomorphic symmetric designs belonging to the Hadamard family of symmetric designs as well as to the family of symmetric designs with parameters of the point-hyperplane design of a projective geometry. For the family of symmetric designs with parameter set

$$v = n^2(n+2), k = n(n+1), \lambda = n; n \geq 2$$

that we just alluded to in the previous paragraph, the author considered a uniform construction of the designs in this class in his paper of 1982 (see [17]). In the same paper, the ideas contained therein were exploited to obtain at least two non-isomorphic symmetric designs for each n. The author is in the process of writing a paper where these ideas are used to show a lower bound $f(n)$ on the number of non-isomorphic symmetric designs with the above parameter set where $f(n)$ is an exponential function of n. Actually $f(n)$ is a constant multiple of the n-th Catalan number (the constant is rather small and hence the bound is large only for large values of prime numbers n).

Quasi-Multiple Designs

If we delete a block of a symmetric design and all the points on that block, the resulting object can be shown to be a design and a design with these parameters is called a *quasi-residual design.* Such a design is called a *residual design* if one can add a suitable set of new points and add suitable new points to the existing blocks so as to obtain a symmetric design. It is easy and also canonical to show that a quasi-residual design with $\lambda = 1$ is a residual design (in fact such a design is just an affine plane) and a result known as the Hall-Connor embedding theorem established the validity of that assertion for $\lambda = 2$. This is not true for higher values of λ; there are counterexamples and recently a large number of simple infinite classes of counter examples was produced by Fleming (see [1]). As an example, we know, using the Bruck-Ryser-Chowla theorem that a biplane (symmetric design with $\lambda = 2$) with parameter set $(22, 7, 2)$ does not exist and hence the Hall-Connor embedding theorem tells us that a (quasi-residual) design with parameter set v, b, r, k, λ equal to $15, 21, 7, 5, 2$ can not exist. However, it may be possible to obtain a design that has m times as many blocks (certainly m can not equal 1). Such a design which is parametrically a multiple of an existent or a non-existent design is called a *quasi-multiple design.* The smallest possible m is called the *index.* It was shown in a paper of the author and S.S. Shrikhande [21] that this particular index for the above parameter set is 2.

The same question could be asked for symmetric designs. For example, we know that a biplane with parameter set $(22, 7, 2)$ does not exist. However, it is possible to construct an 8-fold multiple of this and with some additional properties, one can show (actually shown by Tonchev) that such a design is *unique.* We will return to this question after introducing quasi-symmetric designs in the third section.

Quasi-Symmetric Designs

A design is called a *quasi-symmetric design* if every pair of blocks of the design intersect in x or y points where x and y are constants that depend only on the design. We assume that $x \leq y$. The characterization theorem stated in the first section implies that $x = y$ happens iff the design is a symmetric design. The interest in quasi-symmetric designs stems from the fact that one can draw a graph with vertex set the set of blocks of the design with two vertices joined iff the corresponding blocks intersect in x points. It was shown by S.S. Shrikhande and Bhagwandas (see [22]) that this graph is a *strongly regular graph.* That is, we have a regular graph in which the number of vertices to which two given vertices are commonly adjacent depends only on whether the two vertices are adjacent or not adjacent. Only few classes of quasi-symmetric designs seem to be known. It is known that there is a unique quasi-symmetric 4-design and a similar classification theorem is believed to exist for 3-designs. A Ph.D. thesis of R.M. Pawale [11] obtained a number of results in that direction. In particular, the classification of quasi-symmetric triangle-free 3-designs is known completely (see [12]) and Pawale in the thesis [11] has also shown that no quasi-symmetric 3-design exists for x between 2 to 100.

There are two classical ways of obtaining a quasi-symmetric design.

Example 3.1. Consider an n dimensional projective geometry $PG(n, q)$ where $n \geq 3$. Taking as points, the points of $PG(n, q)$ and as blocks all the subspaces of co-dimension 2 gives a quasi-symmetric design with the parameter set:

$$v = \frac{q^{n+1}-1}{q-1}, \quad k = \frac{q^{n-1}-1}{q-1}, \quad \lambda = \frac{q^{n-1}-1}{q-1},$$
$$x = \frac{q^{n-3}-1}{q-1}, \quad y = \frac{q^{n-2}-1}{q-1}$$

Example 3.2. Let E be a biplane, i.e., a symmetric design with $\lambda = 2$. Then the residual of E is a quasi-symmetric design D with parameter set

$$v' = v - k, \quad k' = k - 2, \quad \lambda' = 2, \quad x = 1, \quad y = 2$$

In fact, an embedding result of Hall and Connor that was referred to earlier asserts that any quasi-symmetric design D with parameters as in Example 3.2 is uniquely obtained from a corresponding biplane E exactly as described in the construction.

Unfortunately, unlike symmetric designs, not too many families of quasi-symmetric designs are known to exist. An example other than the one described above is that of an affine resolvable design. It was proved by Pawale and the author as well as Calderbank and Morton (see [2, 10]) that

Theorem 3.3. *Up to isomorphism, the only quasi-symmetric* 3*-design with* $(x, y) = (1, 3)$ *are the unique* $4-(23, 7, 1)$*-design or its block residual.*

One expects that more quasi-symmetric designs become available in time to come. This question also relates to the determination of strongly regular graphs in an obvious manner. Quasi-symmetric designs are also believed to be connected (in some way) with symmetric designs though the precise connection remains a mystery. The last section is devoted to two recent results of the author in this direction.

Quasi-Multiple Quasi-Symmetric Designs

In earlier sections, we mentioned that for a large number of parameter sets (v, k, λ), the existence of a symmetric design is undecided. Even in situations where a symmetric (v, k, λ)-design does exist, it is worth while looking at parameter sets of the following kind:

$$v' = v, b' = sv, r' = sk, k' = k, \lambda' = \lambda$$

These designs were mentioned in the second section and such a design is called an *s-quasi-multiple* (of a symmetric (v, k, λ)-design), where s is an integer, $s \geq 1$. If such a design is actually constructed by taking s copies of a symmetric (v, k, λ)-design, then is called a *multiple* design. By definition, *proper* quasi-multiple designs are those that are not obtained by taking copies of a symmetric design.

Observe also that the existence question of an s-quasi-multiple design makes sense irrespective of whether a corresponding symmetric design exists or not. But this question naturally assumes a lot of importance when a corresponding symmetric design does not exist (is known not to exist). It is obvious that quasi-multiple designs with particular parameters can be constructed using suitable subsets of a finite field. For example, a $(7, 3, 1)$ symmetric design is construted by taking quadratic residues in the field with 7 elements. Similarly, a $(37, 9, 2)$ symmetric design is constructed by taking biquadratic residues in the field $GF(37)$ (see Beth et al. [1]). In general, suppose $v = \frac{k(k-1)}{2} + 1$ is such that a $GF(v)$ exists (necessarily then v is a prime power). Now suppose $u = \frac{k-1}{2}$ is an integer. Now consider the "development" of sets of the following kind in the sense of C.R. Rao or R.C. Bose:

$$A_i = \{\alpha^i, \alpha^{u+i}, \ldots, \alpha^{(k-1)u+i}\}$$

where α is a primitive element of the field. Then by taking sufficiently many A_i's and their developments, one will have constructed a quasi-multiple of a biplane.

In [9], Jungnickel and Tonchev considered proper quasi-multiples (of symmetric designs) which are also *quasi-symmetric*. Let us call such a design *a special design*. Perhaps the existence question of special designs will shed some light on the connection between symmetric and quasi-symmetric designs. However, only one example of a special design seems to be known, the unique design with 22 points and 176 blocks which is a proper 8-fold quasi-multiple of a non-existent $(22, 7, 2)$ symmetric design. In that paper of Jungnickel and Tonchev [9], various results on special designs were proved. In particular, the authors proved that no such special designs which are quasi-multiples of projective planes exist. In [19], I proved a classification result for special designs corresponding to biplanes, extending the results in [9]. Jungnickel and Tonchev also made the following conjecture in [9].

Jungnickel and Tonchev Conjecture. Let D be a s-fold quasi-multiple design with parameters $v, sv, sk, k, s\lambda$. Let D be also a quasi-symmetric design. Further assume that the parameters satisfy $(k, (s-1)\lambda) = 1$. (here (m, n) denotes the g.c.d. of m and n). Then D cannot be proper. That is, D **must be a multiple of a symmetric design.**

Results were proved in [9] by Jungnickel and Tonchev that partially supported the conjecture. More results were obtained by Ionin and M.S. Shrikhande [7] to show that the conjecture is valid under some numerical conditions extending the results proved by Jungnickel and Tonchev. We end this survey with the following. *Recently, in a paper to appear* [20], *I have proved the validity of the Jungnickel-Tonchev conjecture.*

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On T-core Partitions and Quadratic Forms

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We derive a recurrence for $c_t(n)$, the number of t-core partitions of n, where $t \geq 4$. Use of the recurrence requires the solution of a Diophantine equation involving a quadratic form in $t - 2[\frac{t}{3}]$ variables.

1. Introduction

Let a partition, $[\lambda]$, of n be given by: $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. Then the Ferrers-Young diagram of $[\lambda]$ consists of k rows, where the i^{th} row has λ_i nodes, for $1 \leq i \leq k$. The (i, j) hook is the union of the set of nodes (i, k) with $k \geq j$ and the set of nodes (k, j) with $k \geq i$. The *hook number* corresponding to a given hook is the number of nodes in the hook. Let the integer $t \geq 2$. We say that $[\lambda]$ is *t-core* if no hook number is a multiple of t. In the last 20 years, several researchers have investigated t-core partitions. (See [2], [3], [5], [7], [8], [9], [12], [13], [14], [15] and [16].) Let $c_t(n)$ denote the number of t-core partitions of n (with $c_t(0) = 1$). Note that some authors prefer the notation $a_t(n)$. A closed-form formula for $c_3(n)$ that requires the canonical factorization of $3n + 1$ was stated by Klyachko [9], proved by Granville & Ono [3] using the theory of modular forms, and proved independently by the author using combinatorial methods [16]. Hirschhorn & Sellers [5] discovered numerous identities concerning $c_4(n)$. Ono & Sze [15] obtained a closed-form formula for $c_4(n)$ that is valid when $8n + 5$ is squarefree. Klyackho [9] obtained closed-form formulas for $c_5(n)$ and $c_7(n)$ that require the factorization of $n + 1$, $n + 2$ respectively. The latter formulas also appear in a paper by Garvan, Kim, & Stanton [2]. In [16], we presented recursive formulas for $c_4(n)$ and $c_5(n)$. Kachikidze [7], [8] found closed-form formulas for $c_t(n)$ when $t = 11,\ 13,\ 17$.

In this note, in Theorem 1 below, we make maximal use of an identity of Jacobi (namely (3) below) to derive an Euler-type recurrence for $c_t(n)$. This straightforward method yields a global result: our recurrence makes possible the computation of $c_t(n)$ for *all* t, not just for a limited number of prime values of t. Furthermore, in contrast to results obtained using modular forms, this method does not require the factorization of $n + \frac{t^2-1}{24}$.

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If $t \geq 4$, let $u = [\frac{t}{3}]$, $v = t - 3u$, so that $0 \leq v \leq 2$. Let

$$f_t(n) = c_t(n) + \sum_{k \geq 1} (-1)^k (c_t(n - \omega(k)) + c_t(n - \omega(-k)))$$

Then $f_t(n) = 0$ except when $n = tr$ and r is the sum of u triangular numbers and v pentagonal numbers. The latter condition on r, namely equation (6) below, leads to a Diophantine equation that contains a quadratic form in $u + v$ variables. We discuss in detail the cases: $4 \leq t \leq 13$. The cases $9 \leq t \leq 11$ involve class numbers of imaginary quadratic fields.

2. Preliminaries

Definition 1. Let $\omega(n) = \frac{n(3n-1)}{2}$ (pentagonal numbers).

Definition 2. Let $\sigma(n) = \sum_{d|n} d$.

Definition 3. Let $(\frac{a}{p})$ denote the Legendre symbol.

Definition 4. Let

$$\beta(n) = \sum_{d|n} \left(\frac{d}{3}\right) \text{ if } 3 \nmid n$$

Definition 5. If $n \neq s^2$, let $h(-n)$ be the class number of $Q(\sqrt{-n})$.

$$\sum_{n \geq 0} c_t(n) x^n = \prod_{n \geq 1} \frac{(1 - x^{tn})^t}{1 - x^n} \tag{1}$$

$$\prod_{n \geq 1} (1 - x^n) = 1 + \sum_{k \geq 1} (-1)^k (x^{\omega(k)} + x^{\omega(-k)}) \tag{2}$$

$$\prod_{n \geq 1} (1 - x^n)^3 = \sum_{k \geq 0} (-1)^k (2k + 1) x^{\frac{k(k+1)}{2}} \tag{3}$$

If $l \geq 0$ and m is odd, let $g(l, m)$ denote the number of integer solutions of the equation:

$$x^2 + 3y^2 = 2^l m$$

Then

$$g(l,m) = \begin{cases} 0 & \text{if } 2\not| l \\ 2\beta(m) & \text{if } l = 0 \\ 6\beta(m) & \text{if } l > 0, 2|l \end{cases} \tag{4}$$

Remarks. In (1), (2), and (3), we have $|x| < 1$. (1) is the generating function identity for t-core partitions. (2) is due to Euler, (3) to Jacobi. (4) appears in [6, p. 308, Exercise 3].

3. The Main Results

Theorem 1. *If* $t \geq 4$, *let* $u = [\frac{t}{3}]$, $v = t - 3u$. *Let*

$$f_t(n) = c_t(n) + \sum_{k\geq 1}(-1)^k(c_t(n - \omega(k)) + c_t(n - \omega(-k))). \tag{5}$$

If

$$\frac{n}{t} = r = \sum_{i=1}^{u} \frac{a_i(a_i+1)}{2} + \sum_{j=1}^{v} \omega(\pm b_j) \tag{6}$$

then

$$f_t(n) = \sum_{i,j}(-1)^{\sum_{i=1}^{u} a_i + \sum_{j=1}^{v} b_j} \prod_{i=1}^{u}(2a_i + 1) \tag{7}$$

Otherwise, $f_t(n) = 0$.

Remark. If $v = 0$, then the sums involving v vanish from (6) and (7).

Proof. By hypothesis and (1), we have

$$\left(\sum_{n\geq 0} c_t(n)x^n\right)\left(\prod_{n\geq 1}(1 - x^n)\right) = \left(\prod_{n\geq 1}(1 - x^{tn})^3\right)^u \left(\prod_{n\geq 1}(1 - x^{tn})\right)^v \tag{8}$$

Using (2) and (3), we obtain

$$\begin{aligned} \sum_{n\geq 0} f_t(n)x^n &= \left(\sum_{n\geq 0} c_t(n)x^n\right)\prod_{n\geq 1}(1 - x^n) \\ &= \left(\sum_{n\geq 0}(-1)^n(2n+1)x^{\frac{tn(n+1)}{2}}\right)^u \left(1 + \sum_{n\geq 1}(-1)^n(x^{t\omega(n)} + x^{t\omega(-n)})\right)^v \end{aligned}$$

The conclusion follows from matching coefficients of like powers of x. □

Remark. The right member of (8) is a power series in x^t; the coefficient of x^{tr} is zero unless r is a sum of u triangular numbers and v pentagonal numbers, that is

$$r = \sum_{j=1}^{v} \frac{b_j(3b_j \pm 1)}{2} + \sum_{i=1}^{u} \frac{a_i(a_i + 1)}{2} \tag{9}$$

Let $\psi_t(r)$ denote the number of solutions of (9) in non-negative integers. (Solutions that differ only in the order of terms are considered distinct.) Letting $x_j = 6b_j \pm 1$, $y_i = 2a_i + 1$, we get

$$\sum_{j=1}^{v} x_j^2 + 3\sum_{i=1}^{u} y_i^2 = 24r + 3u + v \tag{10}$$

If $v = 0$, then (10) simplifies to:

$$\sum_{i=1}^{u} y_i^2 = 8r + u \tag{11}$$

Note that in (10) and (11), we have $(x_j, 6) = (y_i, 2) = 1$ for all i, j.

We next consider in detail the cases: $4 \leq t \leq 13$. In each case, we find a formula for $\psi_t(r)$. (Before solving any equation, it is useful to know how many solutions are anticipated.) The proofs of Theorems 2, 5, 7, 9, 11, 13, 15, 17, 19, and 21 are omitted, since each of these theorems is a special case of Theorem 1.

4. The cases: $4 \leq t \leq 13$

If $t = 4$, then we have

Theorem 2.

$$\begin{aligned} &c_4(n) + \sum_{k\geq 1}(-1)^k(c_4(n - \omega(k)) + c_4(n - \omega(-k))) \\ &= \begin{cases} \sum_{a,b}(-1)^{a+b}(2b + 1) & \textit{if } n = 4r, r = \frac{a(3a\pm1)}{2} + \frac{b(b+1)}{2} \\ 0 & \textit{otherwise} \end{cases} \end{aligned} \tag{12}$$

Remark. We have $u = v = 1$, so let $x = 6a \pm 1$, $y = 2b + 1$. The exceptional values of n that arise in (12) are those such that the equation:

$$x^2 + 3y^2 = 4(6r + 1) \tag{13}$$

has solutions in positive integers x, y such that $(x, 6) = (y, 2) = 1$.

Theorem 3.

$$\psi_4(r) = \beta(6r+1) \tag{14}$$

Proof. According to (4), the number of solutions of (13) is $6\beta(6r+1)$. Clearly, (13) implies $x \equiv y \pmod 2$. If $x \equiv y \equiv 0 \pmod 2$, then (13) implies

$$\left(\frac{x}{2}\right)^2 + 3\left(\frac{y}{2}\right)^2 = 6r+1 \tag{15}$$

According to (4), (15) has $2\beta(6r+1)$ solutions. Therefore (13) has $4\beta(6r+1)$ solutions such that x and y are both odd. Also, (13) implies that $(x,3) = 1$. The conclusion follows if we also insist that $x > 0$, $y > 0$. □

Theorem 4 below facilitates the computation of $\psi_4(r)$.

Theorem 4. *Let*

$$m = \prod_{i=1}^{h} p_i^{e_i} \prod_{j=1}^{k} q_j^{f_j} \tag{16}$$

where each prime p_i *satisfies:* $p_i \equiv 1 \pmod 3$ *and each prime* q_j *satisfies:* $q_j \equiv 2 \pmod 3$. *Then*

$$\beta(m) = \begin{cases} \prod_{i=1}^{h}(e_i+1) & \textit{if all } f_j \equiv 0 \pmod 2 \\ 0 & \textit{otherwise} \end{cases} \tag{17}$$

Proof. The conclusion follows from the easily verfied facts that

(i) β is a multiplicative function; (ii) if p is prime and $s \geq 1$, then

$$\beta(p^s) = \begin{cases} s+1 & \text{if } p \equiv 1 \pmod 3 \\ \frac{1+(-1)^s}{2} & \text{if } p \equiv 2 \pmod 3 \end{cases} \tag{18}$$

□

Remark. There are infinitely many integers, r, such that $\psi_4(r) = 0$, namely, $r = \frac{pq-1}{6}$, where p, q are distinct primes such that $p \equiv q \equiv 2 \pmod 3$.

If $t = 5$, then we have:

Theorem 5.

$$c_5(n) + \sum_{k\geq 1}(c_5(n-\omega(k)) + c_5(n-\omega(-k)))$$

$$= \begin{cases} \sum_{a,b,c}(-1)^{a+b+c}(2c+1) & \textit{if } n = 5r, r = \frac{a(3a\pm1)+b(3b\pm1)+c(c+1)}{2} \\ 0 & \textit{otherwise} \end{cases} \tag{19}$$

Remark. We have $u = 1$, $v = 2$, so let $x = 6a \pm 1$, $y = 6b \pm 1$, $z = 2c + 1$.

Theorem 6.

$$\psi_5(r) = \sum_{(x,6)=1} \beta\left(6r + \frac{5-x^2}{4}\right) \tag{20}$$

Proof. We seek positive integer solutions of the equation:

$$x^2 + y^2 + 3z^2 = 24r + 5 \tag{21}$$

such that $(x, 6) = (y, 6) = (z, 2) = 1$. Let us rewrite (21) as

$$y^2 + 3z^2 = 4\left(6r + \frac{5-x^2}{4}\right) \tag{22}$$

For a given integer, x, such that $(x, 6) = 1$, referring to (4) and reasoning as in the proof of Theorem 3, we have $\beta(6r + \frac{5-x^2}{4})$ solutions of (22) in odd, positive integers y, z. The conclusion now follows. □

If $t = 6$, then we have:

Theorem 7.

$$\begin{aligned} &c_6(n) + \sum_{k\geq 1}(-1)^k(c_6(n - \omega(k)) + c_6(n - \omega(-k))) \\ &= \begin{cases} \sum_{a,b}(-1)^{a+b}(2a+1)(2b+1) & \textit{if } \frac{n}{6} = r = \frac{a(a+1)+b(b+1)}{2} \\ 0 & \textit{otherwise} \end{cases} \end{aligned} \tag{23}$$

Remark. We have $u = 2$, $v = 0$, so let $x = 2a + 1$, $y = 2b + 1$.

Theorem 8. *Let*

$$4r + 1 = \prod_{i=1}^{h} p_i^{e_i} \prod_{j=1} q_j^{f_j} \tag{24}$$

where each prime p_i satisfies: $p_i \equiv 1 \pmod 4$ and each prime q_j satisfies: $q_j \equiv 3 \pmod 4$ then

$$\psi_6(r) = \begin{cases} \prod_{i=1}^{h}(e_i + 1) & \textit{if all } f_j \equiv 0 \pmod 2 \\ 0 & \textit{otherwise} \end{cases} \tag{25}$$

Proof. The exceptional values of n that arise in (23) correspond to solutions of:

$$x^2 + y^2 = 2(4r + 1) \tag{26}$$

The conclusion follows from [11, Theorem 3.22, p. 166]. ☐

If $t = 7$, we have:

Theorem 9. If

$$\frac{n}{7} = r = \frac{1}{2}(a(3a \pm 1) + b(b + 1) + c(c + 1)) \tag{27}$$

then

$$f_7(n) = \sum_{a,b,c} (-1)^{a+b+c}(2b + 1)(2c + 1)$$

Otherwise, $f_7(n) = 0$.

Theorem 10.

$$\psi_7(r) = \sum_{x \equiv 3 \pmod 6} \beta\left(18r + \frac{21 - x^2}{4}\right) \tag{28}$$

Proof. let $x = 6b + 3$, $y = 6c + 3$, $z = 6a \pm 1$, so that the exceptional values of n that arise in (27) are solutions of

$$x^2 + y^2 + 3z^2 = 72r + 21 \tag{29}$$

where $x \equiv y \equiv 3 \pmod 6$ and $z \equiv 1 \pmod 2$. We may rewrite (28) as

$$y^2 + 3z^2 = 4\left(18r + \frac{21 - x^2}{4}\right) \tag{30}$$

Let x be a given positive integer such that $x \equiv 3 \pmod 6$.

Referring to (4) and reasoning as in the proof of Theorem 3, we have $\beta(18r + \frac{21-x^2}{4})$ solutions of (30) in odd, positive integers y, z. The conclusion now follows. ☐

If $t = 8$, we have:

Theorem 11. *If*

$$\frac{n}{8} = r = \frac{a(3a \pm 1) + b(3b \pm 1) + c(c + 1) + d(d + 1)}{2} \tag{31}$$

then

$$f_8(n) = \sum_{a,b,c,d} (-1)^{a+b+c+d}(2c+1)(2d+1)$$

Otherwise, $f_8(n) = 0$.

Theorem 12.

$$\psi_8(r) = \sum_{i,j\geq 0, i+j=r} \beta(6i+1)\beta(6j+1) \tag{32}$$

Proof. Let $x = 6a \pm 1$, $y = 2c+1$, $z = 6b \pm 1$, $w = 2d+1$, so that the exceptional values of n that arise in (31) are solutions of:

$$x^2 + 3y^2 + z^2 + 3w^2 = 24r + 8 \tag{33}$$

Since $x^2 \equiv z^2 \equiv 1 \pmod{24}$ and $y^2 \equiv w^2 \equiv 1 \pmod{8}$, we have $x^2 + 3y^2 \equiv z^2 + 3w^2 \equiv 4 \pmod{24}$, that is,

$$x^2 + 3y^2 = 24i + 4 \tag{34}$$

and

$$z^2 + 3w^2 = 24j + 4 \tag{35}$$

where $i \geq 0$, $j \geq 0$. Furthermore, (33) implies that $i + j = r$. Since the solutions of (34) and (35) may be chosen independently, the conclusion now follows from Theorem 3. □

If $t = 9$, then we have:

Theorem 13. *If*

$$\frac{n}{9} = r = \frac{a(a+1) + b(b+1) + c(c+1)}{2} \tag{36}$$

then

$$f_9(n) = \sum_{a,b,c} (2a+1)(2b+1)(2c+1)$$

Otherwise, $f_9(n) = 0$.

Theorem 14.

$$\psi_9(n) = 3\left(\sum_{d^2|(8r+3)} h\left(-\frac{8r+3}{d^2}\right)\right) + j \tag{37}$$

where

$$j = \begin{cases} -2 & \text{if } 8r+3 = 3k^2 \\ 0 & \text{otherwise} \end{cases} \tag{38}$$

Proof. Let $x = 2a+1$, $y = 2b+1$, $z = 2c+1$, so that the exceptional values of r that arise in (36) are the solutions in positive integers of:

$$x^2 + y^2 + z^2 = 8r + 3 \tag{39}$$

The conclusion now follows from [1, Theorem 8.7, p. 178]. □

If $t = 10$, then we have:

Theorem 15. *If*

$$\frac{n}{10} = r = \frac{a(a+1) + b(b+1) + c(c+1) + d(3d \pm 1)}{2} \tag{40}$$

then

$$f_{10}(n) = (-1)^{a+b+c+d}(2a+1)(2b+1)(2c+1)$$

Otherwise, $f_{10}(n) = 0$.

Theorem 16.

$$\psi_{10}(r) = \sum_{(m,6)=1} \left\{ 3 \sum_{d^2 | (8r+3+\frac{1-m^2}{3})} h\left(-\frac{8r+3+\frac{1-m^2}{3}}{d^2} \right) \right\} + \epsilon_r \tag{41}$$

where

$$\epsilon_r = \begin{cases} -2 & \text{if } 8r+3+\frac{1-m^2}{3} = 3k^2 \\ 0 & \text{otherwise} \end{cases} \tag{42}$$

Proof. Let $x = 2a+1$, $y = 2b+1$, $z = 2c+1$, $m = 6d \pm 1$, so that the exceptional values of n that arise in (40) are the solutions in odd, positive integers of:

$$x^2 + y^2 + z^2 = 8r + 3 + \frac{1-m^2}{3} \tag{43}$$

By Theorem 14, we have:

$$\psi_{10}(r) = \sum_{(m,6)=1} \psi_9\left(r + \frac{1-m^2}{24}\right) \tag{44}$$

from which the conclusion follows. $\square$

If $t = 11$, we have:

Theorem 17. *If*

$$\frac{n}{11} = r = \frac{\sum_{i=1}^{3} a_i(a_i+1) + \sum_{i=4}^{5} a_i(3a_i \pm 1)}{2}$$

then

$$f_{11}(n) = (-1)^{\sum_{i=1}^{5} a_i} \prod_{i=1}^{3}(2a_i+1)$$

Otherwise, $f_{11}(n) = 0$.

Theorem 18.

$$\psi_{11}(r) = \sum_{(j,6)=1} \sum_{(k,6)=1} \left\{ \sum_{d^2 \mid (8r+3+\frac{1-j^2-k^2}{3})} 3h\left(-\frac{8r+3+\frac{1-j^2-k^2}{3}}{d^2}\right) + \epsilon_{jk} \right\}$$

(45)

where

$$\epsilon_{jk} = \begin{cases} -2 & \text{if } 8r+3+\frac{1-j^2-k^2}{3} = 3s^2 \\ 0 & \text{otherwise} \end{cases} \tag{46}$$

Proof. The proof is similar to that of Theorem 16, and is thus omitted. $\square$

If $t = 12$, we have:

Theorem 19. *If*

$$\frac{n}{12} = r = \frac{a(a+1) + b(b+1) + c(c+1) + d(d+1)}{2} \tag{47}$$

then

$$f_{12}(n) = (-1)^{a+b+c+d}(2a+1)(2b+1)2c+1)(2d+1)$$

Otherwise, $f_{12}(n) = 0$.

Theorem 20.

$$\psi_{12}(r) = \sigma(2r+1) \tag{48}$$

Proof. Let $x = 2a+1$, $y = 2b+1$, $z = 2c+1$, $w = 2d+1$, so that the exceptional values of n that arise in (47) are the solutions in odd, positive integers of:

$$x^2 + y^2 + z^2 + w^2 = 8r + 4 \tag{49}$$

According to [4, Theorem 286, p. 314], if $r_4(m)$ is the number of representations of m as a a sum of four squares, then

$$r_4(m) = 8 \sum \{d : d|m, 4 \nmid d\} \tag{50}$$

Thus we have:

$$r_4(8r+4) = 8\sigma(4r+2) = 24\sigma(2r+1) \tag{51}$$

Now the number of representations of $8r+4$ as a sum of four even squares is $r_4(2r+1) = 8\sigma(2r+1)$. Clearly, in (50), we have $x \equiv y \equiv z \equiv w \pmod 2$. Therefore $8r+4$ has $16\sigma(2r+1)$ representations as a sum of four odd squares. The conclusion follows if we insist that x, y, z, w all be positive. □

If $t = 13$, we have:

Theorem 21.

$$\begin{aligned} &c_{13}(n) + \sum_{k \geq 1} c_{13}(n - \omega(k) + c_{13}(n - \omega(-k))) \\ &= \begin{cases} (-1)^{\sum_{i=1}^{5} a_i} \prod_{i=1}^{4}(2a_i+1) & \textit{if } \frac{n}{13} = r = \frac{a_5(3a_5 \pm 1) + \sum_{i=1}^{4} a_i(a_i+1)}{2} \\ 0 & \textit{otherwise} \end{cases} \end{aligned} \tag{52}$$

Theorem 22.

$$\psi_{13}(r) = \sum_{(m,6)=1} \sigma\left(2r+1+\frac{1-m^2}{12}\right) \tag{53}$$

Proof. The proof is similar to that of Theorem 16, and is hence omitted. □

Concluding Remarks. Some recent results by Stephen C. Milne [10], when suitably adapted, might facilitate the computation of $\psi_t(n)$ when $t = 12k^2$ or $t = 12k(k+1)$.

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Observations on Some Algebraic Equations Associated with Ramanujan's Work

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This note collects different observations stimulated by Ramanujan's results on algebraic equations, namely, the equation with 3 iterated square roots $x = \sqrt{t + \sqrt{t + \sqrt{t + x}}}$, which he solved by radicals, and the Diophantine equation $x^3 + y^3 + z^3 = 1$, which appears in *The Lost Notebook*, along with an astonishing solution. It is shown that, in general, the equation with 5 iterated square roots cannot be solved by radicals and that the Diophantine equation has solutions not previously quoted.

1. Introduction

The occurrence of radicals in Ramanujan's work has been discussed in many papers in which his results have been confirmed using techniques probably different from his original methods. The aim of this paper is to collect some observations stimulated by Ramanujan's results on two algebraic equations, in particular:

i) It is proved that the equation

$$x = \pm\sqrt{t \pm \sqrt{t \pm \sqrt{t \pm \sqrt{t \pm \sqrt{t + x}}}}} \tag{1}$$

cannot be solved using radicals when t is transcendental over $\mathbb{Q}$. Using radicals, solutions of (1) are possible with $t = 0, 2$, and $4/3$. Finding solutions, again using radicals, for other values of t remains an open problem.

ii) In [12], Hirschorn offers a proof of Ramanujan's solution of the Diophantine equation

$$x^3 + y^3 + z^3 = 1 \tag{2}$$

from *The Lost Notebook* [1]. Hirschorn's method yields solutions not previously quoted.

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Equation (1) is easily seen to be equivalent to a cyclic system

$$(3) \qquad \begin{cases} x_1^2 - x_2 - t = 0 \\ x_2^2 - x_3 - t = 0 \\ \quad \dots \\ x_n^2 - x_1 - t = 0 \end{cases}$$

of n quadratic equations in n variables, with $n = 5$. This system can be considered as a Diophantine system of n equations and $n + 1$ variables, where integer and rational solutions are sought after. Therefore, it is shown that with $n = 3$, a rational (parametric) solution exists, but integer solutions do not exist.

2. Cyclic Systems

The system in (3) with $n = 3$, proposed by Ramanujan in the *Journal of the Indian Mathematical Society* as Question 507, [2], was solved by Ramanujan [2] himself. The system with $n = 4$ was also proposed by Ramanujan in the same Indian journal [2] as Question 722, and the first solution was given by Rao in [14], where similar questions with five equations were also examined. Other solutions were found by Watson [22], Salam [15], and Berndt [3]. The system with $n = 5$ and $t = 2$ is related to the 11th cyclotomic polynomial and was brilliantly solved by Vandermonde using radicals [19]. The system with $n = 5$ and $t = 0$ is related to the solution (by radicals) of the 31st cyclotomic polynomial [18]. In spite of these results, it is proved that the system in general cannot be solved using radicals.

Let t be transcendental over $\mathbb{Q}$ if not specified otherwise. Let a row vector $\mathbf{X}_j = (x_{1j}, x_{2j}, x_{3j}, x_{4j}, x_{5j})$ denote a solution of the system in (3) with $n = 5$, where

$$(4) \qquad \begin{array}{ll} x_{2j} = x_{1j}^2 - t, & x_{3j} = (x_{1j}^2 - t)^2 - t, \\ x_{4j} = ((x_{1j}^2 - t)^2 - t)^2 - t, & x_{5j} = (((x_{1j}^2 - t)^2 - t)^2 - t)^2 - t. \end{array}$$

The equation $x_{5j}^2 - x_{1j} - t = 0$ implies that x_{1j} and thus, any x_{ij}, $1 \leq i \leq 5$, is a root of the polynomial

$$R_5(x, t) = (((x^2 - t)^2 - t)^2 - t)^2 - t)^2 - x - t.$$

Any cyclic permutation π of the entries in $\mathbf{X}_j$ produces another solution $\pi(\mathbf{X}_j) = (x_{2j}, x_{3j}, x_{4j}, x_{5j}, x_{1j})$, said to be cyclically-equivalent to $\mathbf{X}_j$. If the entries in $\mathbf{X}_j$ are not all equal, the set $S_j = \{\pi^i(\mathbf{X}_j) : 0 \leq i \leq 4\}$ contains 5 distinct solutions.

Theorem 1. *$R_5(x, t)$ splits over $\mathbb{Q}(t)$ into two irreducible factors:*

1) *$F_2(x, t) = x^2 - x - t$ of degree 2, which accounts for two trivial solutions of (3) with $x_1 = x_2 = x_3 = x_4 = x_5$.*
2) *$F_5(x, t) = x^{30} + x^{29} - (15t - 1)x^{28} + \cdots + 1$ of degree 30, which accounts for thirty solutions of (3).*

The roots of $F_5(x,t)$ are partitioned into 6 sets S_j of cyclically-equivalent solutions. Each set S_j is associated to a 5-degree polynomial

$$A_5(x,t,\sigma_{1j}) = \prod_{i=1}^{5}(x - x_{ij}) \qquad j = 1,\ldots,6$$

where $\sigma_{1j} = x_{1j} + x_{2j} + x_{3j} + x_{4j} + x_{5j}$, and

$$F_5(x,t) = \prod_{j=1}^{6} A_5(x,t,\sigma_{1j}) \, .$$

Proof. $F_2(x,t)$ is clearly irreducible, and $x^2 - t \equiv x \bmod F_2(x,t)$. Therefore $R_5(x,t) = 0 \bmod F_2(x,t)$, which means $F_2(x,t)$ divides $R_5(x,t)$.

Let $F_5(x,t)$ denote the quotient $R_5(x,t)/F_2(x,t)$, which can be viewed as a difference polynomial $x^{30} + \sum_{i=1}^{30} p_i(t)x^{30-i}$ of degree 30 in x and degree 15 in t [13]. The degrees of the polynomials $p_i(t)$ in t are non-decreasing when i increases, and $p_{30}(t)$ has the largest degree 15. Therefore, $F_5(x,t)$ is irreducible by either Theorem 1.1 in [16] or Theorem 6 in [13]. However, an elementary short proof of $F_5(x,t)$ irreducibility based on a specialization principle [17], is possible. Since $F_5(x,0)$ is the monic 31st cyclotomic polynomial, and it is irreducible [9, p. 154], a fortiori $F_5(x,t)$ is irreducible over $\mathbb{Q}(t)$.

The remaining properties are an immediate consequence of the definition of cyclically-equivalent solutions. □

In view of Theorem 1, to solve for $R_5(x,t)$ is equivalent to solving for $F_5(x,t)$. In the following, it will be shown that $F_5(x,t)$ cannot be solved using radicals, thus, $R_5(x,t)$ cannot be solved as well. Two preliminary lemmas are required.

Lemma 1. *The Galois group $\mathfrak{Gal}(F_5(x,t))$ of $F_5(x,t)$ over $\mathbb{Q}(t)$ is transitive. The Galois group $\mathfrak{Gal}(A_5(x,t))$ of each factor $A_5(x,t,\sigma_{1j})$ over the respective coefficient field $\mathbb{L}_j = \mathbb{Q}(t,\sigma_{1j})$ is either cyclic of order 5 or the identity group. In both cases, the equation $A_5(x,t,\sigma_{1j}) = 0$ is abelian.*

Proof. $\mathfrak{Gal}(F_5(x,t))$ is obviously transitive because $F_5(x,t)$ is irreducible over $\mathbb{Q}(t)$. If $A_5(x,t,\sigma_{1j})$ is irreducible in $\mathbb{L}_j$, its Galois group [8, p. 70] is cyclic of order 5 because Tschirnhaus's transformation[1] $y = -t + x^2$ cyclically permutes its roots [6]. If $A_5(x,t,\sigma_{1j})$ splits in $\mathbb{L}_j$, its Galois group is the identity group. In both cases, the equation $A_5(x,t,\sigma_{1j}) = 0$ is abelian. □

The factorization of $F_5(x,t)$ into 5-degree polynomials is based on the Galois-Lagrange resolvent $L_5(x,t)$ [23], a root of which generates $\mathbb{L}_j$. Many methods have been devised to compute Galois-Lagrange resolvents [6, 18, 8] for different situations. Any method considers a convenient Galois-Lagrange resolvent function. In this context, a possible Galois-Lagrange resolvent function is σ_{1j}, which is a root of $L_5(x,t)$. This polynomial is computed solving a linear system of equations.

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[1] Recall that the most general Tschirnhaus transformation of a degree n polynomial $p(x)$ is given by a polynomial of degree $n-1$ of the form $y = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$.

Lemma 2. *Let* σ_{1j} *be the Galois-Lagrange resolvent function for* $F_5(x,t)$*. The corresponding Galois-Lagrange resolvent*

$$L_5(x,t) = x^6 - \lambda_1 x^5 + \lambda_2 x^4 - \lambda_3 x^3 + \lambda_4 x^2 - \lambda_5 x + \lambda_6$$

is computed solving a linear system of equations obtained from the polynomial identity

$$L_5(S_5(x,t),t) \equiv 0 \mod F_5(x,t), \tag{5}$$

where

$$\begin{aligned} S_5(x,t) = {} & x + (x^2-t) + ((x^2-t)^2-t) + (((x^2-t)^2-t)^2-t) \\ & + ((((x^2-t)^2-t)^2-t)^2-t). \end{aligned}$$

Proof. Let α denote a root of $A_5(x,t,\sigma_{1j})$, then $F_5(\alpha,t) = 0$, $\sigma_{1j} = S_5(\alpha,t)$, and $L_5(S_5(\alpha,t),t) = 0$ by assumption. Therefore, a Lagrange theorem [9, p. 57] implies $F_5(x,t) | L_5(S_5(x,t),t)$ or equivalently

$$L_5(S_5(x,t),t) = 0 \mod F_5(x,t),$$

yielding a linear system of 30 equations in 6 unknowns λ_i, $i = 1,\ldots,6$, which certainly admits a solution by Lemma 1. □

The method for computing $L_5(x,t)$ shown in Lemma 2 applies to every n with obvious modifications. The following Galois-Lagrange resolvents for $n = 2, 3, 4$, and 5 have been computed using MAPLE:

$$\begin{aligned} L_2(x,t) &= x+1, \quad L_3(x,t) = x^2 + x - t + 2, \\ L_4(x,t) &= x^3 - (4t-3)x - 4 \\ L_5(x,t) &= x^6 + x^5 + (3-11t)x^4 + (11-18t)x^3 + (44-19t+19t^2)x^2 \\ &\quad + (36+24t+17t^2)x + 32 - 28t + 40t^2 - 9t^3. \end{aligned}$$

The time-consuming computations occur only when $n = 5$. However, in this case the burden of calculations can be considerably reduced, observing that $\sum_{j=1}^{6} \sigma_{1j} = -1 = \lambda_1$, so that only five independent linear equations are required to find all λs. For example, these equations are obtained from the constant term and the coefficients of x, x^2, x^{26}, and x^{28} in (5).

Theorem 2. *The Galois group $\mathfrak{G}(L_5(x,t))$ of $L_5(x,t)$ with respect to the field $\mathbb{Q}(t)$ is the symmetric group S_6.*

Since S_6 is a non solvable group, then $F_5(x,t) = 0$ and, in turn, $R_5(x,t)$ is not solvable by radicals.

Proof. To prove that $\mathfrak{G}(L_5(x,t))$ is S_6, it is sufficient to show that S_6 is the Galois group with respect to the field $\mathbb{Q}$ of a particular polynomial with a given rational value of t. To this aim, we apply a technique described in [20, vol. I: p. 190] to the polynomial

$$L_5(x,-1) = x^6 + x^5 + 14x^4 + 29x^3 + 82x^2 + 29x + 109.$$

which is irreducible over $\mathbb{Q}$, because it is irreducible modulo 17. This polynomial considered modulo 13 and 1381 respectively, factors as

$$\begin{aligned}&(x^5 + 12x^4 + 3x^3 + 10x^2 + 10x + 9)(x+2) \mod 13\\ &(x^2 + 743x + 745)(x+891)(x+7)(x+1205)(x+1298) \mod 1381.\end{aligned}$$

The occurrence of the factor of degree 5 in the factorization modulo 13 implies that the Galois group of $L_5(x,-1)$ contains a cycle of 5 symbols. The occurrence of the factor of degree 2 as unique non-linear factor in the factorization modulo 1381 implies that the Galois group of $L_5(x,-1)$ contains a cycle of 2 symbols. The conclusion follows from the irreducibility of $L_5(x,-1)$ over $\mathbb{Q}$ and from a theorem [20, vol. I: p. 191] stating that *a transitive permutation group of n objects containing a cycle of two symbols and an $(n-1)$ cycle is a symmetric group* with $n = 6$.

In conclusion, $L_5(x,t) = 0$ cannot be solved by radicals because its Galois group S_6 is a non-solvable group. $F_5(x,t) = 0$ cannot be solved by radicals. □

Given a root σ_{1j} of $L_5(x,t)$, it is also of interest to know the coefficients of

$$A_5(x,t,\sigma_{1j}) = x^5 - \sigma_{1j}x^4 + \sigma_{2j}x^3 - \sigma_{3j}x^2 + \sigma_{4j}x - \sigma_{5j}$$

in terms of σ_{1j}. Since $A_5(x,t,\sigma_{1j})$ is left invariant by Tschirnhaus's transformation $\mathfrak{T}:\ y = -t + x^2$, then we have the identity

$$p_e(x^2) + xp_o(x^2) = p_e(x+t)^2 - (x+t)p_o(x+t)^2, \tag{6}$$

where $p_e(.)$ and $p_o(.)$ are the even and odd part of $A_5(x,\sigma_{1j})$ respectively. From this identity, a system of 5 equations for $\sigma_{1j}, \sigma_{2j}, \sigma_{3j}, \sigma_{4j}$, and σ_{5j} is obtained. Therefore, every coefficient is expressed in terms of σ_{1j} and a 52-degree equation $T_{52}(x) = 0$ for σ_{1j} is finally obtained. But, in our problem, we already know that

σ_{1j} is a root of $L_5(x,t)$, which turns out to be an irreducible factor of $T_{52}(x)$. The coefficients of $A_5(x,t,\sigma_{1j})$ result in:

$$(7)\quad \begin{cases} \sigma_{1j} = a \\ \sigma_{2j} = \frac{a^2-a-5t}{2} \\ \sigma_{3j} = -\frac{a^5+(7-14t)a^3+(2t-4)a^2+(61t^2+28-62t)a+10t^2+20t}{8(3t-4)} \\ \sigma_{4j} = \frac{a^5+(7-14t)a^3+(-12t^2+6t+12)a^2+(25t^2-26t+44)a+36t^3-50t^2+12t+32}{8(3t-4)} \\ \sigma_{5j} = \frac{ta^5-(14t^2-7t)a^3-(10t^2-12t)a^2+(37t^3-42t^2+44t)a+22t^3+56t-20t^2-32}{8(3t-4)} \end{cases}$$

The same technique is applied to find invariant polynomials of every degree n. For later use with $n = 3$, we find

$$A_3(x,t,a) = x^3 - ax^2 - (a+t+1)x - at + 1 - t$$

where a is a root of $L_3(x,t) = x^2 + x + 2 - t$.

The particular polynomial $F_5(x,2)$ has both historical and technical interest. Its Galois-Lagrange resolvent splits over $\mathbb{Q}$ as

$$L_5(x,2) = (x+1)(x^2-x-8)(x^3+x^2-10x-8),$$

thus, $F_5(x,2)$ splits over $\mathbb{Q}$ in three irreducible factors of degree 5, 10, and 15, respectively. The factor of degree 5 is the Vandermonde irreducible quintic $m_5(x) = x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1 = A_5(x,t,-1)$ having a cyclic Galois group of order 5. The roots of $m_5(x)$ were computed by Vandermonde in the search for the eleventh roots of unity. These roots stem from the quadratic equation $x^2 - x_i x + 1 = 0$ where x_i is a root of $m_5(x)$ [18].

The previous remarks, together with Weber's proposition Sazt 3. [23, vol. I, p. 563], show

Corollary 1. *The Galois group of every* $A_5(x,t,\sigma_{1j})$ *is a cyclic group of order* 5.

In terms of extension fields [4], these theorems show that a root α of the irreducible polynomial $F_5(x,t)$ generates an extension field $\mathbb{K} = \mathbb{Q}(t,\alpha)$ of degree 30 of $\mathbb{Q}(t)$. A root β of the irreducible polynomial $L_5(x,t)$, (the Galois-Lagrange resolvent), generates an extension field $\mathbb{L} = \mathbb{Q}(t,\beta)$ of degree 6 of $\mathbb{Q}(t)$. Furthermore, $\mathbb{K} = \mathbb{L}(\gamma)$ is a normal extension of degree 5 of $\mathbb{L}$, generated by a root γ of $A_5(x,t,\sigma_{1j})$, since this polynomial is irreducible in $\mathbb{L}$ with cyclic Galois group of order 5. It is interesting to remark that the splitting field $\mathbb{F}$ of $F_5(x,t)$ is much larger than $\mathbb{K}$. The degree of $\mathbb{F}$ over $\mathbb{Q}(t)$ is equal to the order of the Galois group $\mathfrak{Gal}(F_5(x,t))$ [4, p. 42], which is certainly larger than $5 \cdot 6! = 3600$. In conclusion, since, the splitting field $\mathbb{F}$ does not lie in an extension by radicals, because the intermediate field $\mathbb{L}$ is not an extension by radicals, $F_5(x,t)$ is not solvable by radicals [4, p. 72].

2.1 Solving a quintic with a cyclic Galois group

Let $q(x) = x^5 - \sigma_1 x^4 + \sigma_2 x^3 - \sigma_3 x^2 + \sigma_4 x - \sigma_5$ be an irreducible quintic with cyclic Galois group $\mathfrak{Gal}(q(x))$. Let the action of a generator $\mathfrak{g} \in \mathfrak{Gal}(q(x))$ on the roots x_i of $q(x)$ be defined by $\mathfrak{g}(x_i) = x_{i+1} = -t + x_i^2$. Given a primitive fifth root η of the unity, let us form a Lagrange resolvent

$$\Gamma_j = x_1 + \eta^j \mathfrak{g}(x_1) + \eta^{2j} \mathfrak{g}^2(x_1) + \eta^{3j} \mathfrak{g}^3(x_1) + \eta^{4j} \mathfrak{g}^4(x_1).$$

Notice that $\Gamma_0 = \sigma_1$, the sum of the roots. Since $\mathfrak{g}(\Gamma_j) = \eta^{-j}\Gamma_j$, it follows that Γ_j^5, $j = 1, 2, 3, 4$ belongs to the coefficient field; therefore, it can be expressed integrally in terms of the coefficients of the irreducible quintics. For example, given the polynomial

$$\begin{aligned} V(x) &= x^5 - x^4(-\sqrt{33}/2 + 1/2) - x^3 \\ &\quad + x^2(-3\sqrt{33}/2 + 9/2) - x(-\sqrt{33} + 6) + 1, \end{aligned}$$

which is a factor of $F_5(x, 2)$ and σ_1 a root of $x^2 - x - 8$, the power Γ_1^5 can be computed observing that the polynomial

$$\begin{aligned} \Gamma_1(x)^5 = {} & (x + \eta(x^2 - t) + \eta^2((x^2 - t)^2 - t) \\ & + \eta^3(((x^2 - t)^2 - t)^2 - t) + \eta^4((((x^2 - t)^2 - t)^2 - t)^2 - t))^5 \end{aligned}$$

modulo $V(x)$ yields a constant reminder

$$\begin{aligned} \Gamma_1^5 = {} & 143 + 220\sqrt{5} + \tfrac{\sqrt{33}}{2}(164 + 25\sqrt{5}) \\ & + i\sqrt{2}\sqrt{5 + \sqrt{5}}[55 + \tfrac{215}{8}\sqrt{33} + 55\sqrt{5} + \tfrac{15}{8}\sqrt{5}\sqrt{33}]. \end{aligned}$$

All Γ_j can be obtained in this way. Lastly, x_1 is computed as

$$x_1 = \frac{1}{5}(\Gamma_0 + \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4).$$

The apparently degenerate case $t = 0$ corresponds to the solution of the cyclotomic equation for the 31st root of unity. The cyclotomic equation of degree 30 splits into six factors of degree 5 of the form

$$x^5 - \sigma_1 x^4 + \frac{\sigma_1^2 - \sigma_1}{2} x^3 - \frac{(\sigma_1 - 2)(\sigma_1 - 1)(\sigma_1 + 1)}{8} x^2 + 1$$

where σ_1 is a root of $L_5(x, 0) = x^6 + x^5 + 3x^4 + 11x^3 + 44x^2 + 36x + 32$. This splits into two factors of degree 3 over $\mathbb{Q}(\frac{1+i\sqrt{31}}{2})$:

$$x^3 - x^2 \frac{-1 \pm i\sqrt{31}}{2} - \frac{5 \pm i\sqrt{31}}{2} x - (1 \pm i\sqrt{31}).$$

Correspondingly, $F_5(x,0)$ splits over $\mathbb{Q}(\frac{1+i\sqrt{31}}{2})$ in two irreducible factors of degree 15.

These solutions permit us to write all 16 positive solutions of (1) with $t = 2$ using radicals. For example, the two positive roots of Vandermonde's polynomial yields

$$x = \sqrt{2+\sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2+x}}}}} \longrightarrow x \approx 1.68269$$

$$x = \sqrt{2-\sqrt{2-\sqrt{2-\sqrt{2+\sqrt{2+x}}}}} \longrightarrow x \approx 0.83076.$$

Lastly, in spite of Theorem 2, the cyclic system of 5 equations can still be regarded as solvable by radicals if a solution based on an auxiliary parameter λ is permitted. In fact, we can solve $L_5(\lambda, t) = 0$ for t because $L_5(\lambda, t)$ is of degree 3 in t, and we know that $A_5(x, t, \lambda) = 0$ can be solved for x, which results in x being a function of λ only. Therefore, any solution of $F_5(x, t) = 0$ can be expressed in term of a parameter λ using radicals.

3. Diophantine Cyclic Systems

A cyclic system (3) with $n + 1$ unknowns $x_1, x_2, \ldots, x_n$ and t can be considered as a set of n Diophantine equations. Thus, we can look for integral or rational solutions.

The equation $x^2 = x + t$ has a trivial parametric solution $x = m$, and $t = m^2 - m$, therefore, it has both integral and rational solutions.

The system (3) with two equations also has a parametric solution $x = m$, $y = -m - 1$, and $t = m^2 + m + 1$. Therefore, it has integral and rational solutions, too.

The system with three equations is equivalent to the pair of equations

$$\begin{aligned} & x^3 - \sigma x^2 - (1 + t + \sigma)x + \sigma t + t - 1 = 0 \\ & \sigma^2 + \sigma - t + 2 = 0 \end{aligned} \tag{8}$$

where t can be expressed parametrically as a function of σ. Therefore, the first equation defines a cubic curve

$$x^3 - \sigma x^2 - (\sigma^2 + 2\sigma + 3)x + \sigma^3 + 2\sigma^2 + 3\sigma + 1 = 0. \tag{9}$$

This is a singular cubic with the singular point $(1, 1, 0)$ at infinity and with a point (ξ_1, ξ_2, ξ_0) in homogeneous coordinates $x = \xi_1/\xi_0$ and $\sigma = \xi_2/\xi_0$. This cubic is a rational curve [21, Theorem 5.1, p. 67] whose equation with the change of variables $x = (u + v)/2$ and $\sigma = (u - v)/2$ assumes the form

$$uv^2 - uv + v^2 - 3v + 1 = 0\,. \tag{10}$$

From this expression, it is straightforward to obtain the parametric solution of system (3) with $n = 3$

$$\begin{aligned} x_1 &= \frac{v^3 - 2v^2 + 3v - 1}{2v(v-1)}, & x_2 &= -\frac{v^3 - 2v^2 + v - 1}{2v(v-1)} \\ x_3 &= -\frac{v^3 - v + 1}{2v(v-1)}, & t &= \frac{v^6 - 2v^5 + 4v^4 - 8v^3 + 9v^2 - 4v + 1}{4v^2(v-1)^2}. \end{aligned}$$

Therefore, the system with 3 equations has rational solutions, but no integral solutions are possible. In fact, if t and x are integers, then based on the first equation (8), σ is a rational number. A rational number which is a root of a monic polynomial, namely, the second equation (8), is an integer. Therefore, with $t = 2 + \sigma + \sigma^2$, x is a root of the equation (9). The substitution $x = (u + v)/2$ and $\sigma = (u - v)/2$ leads to (10), which yields

$$u = \frac{v^2 - 3v + 1}{v(1 - v)}.$$

This condition shows that u is not an integer because $v^2 - 3v + 1$ is relatively prime with v and $(v - 1)$. Since $u = x + \sigma$ is an integer, the contradiction proves that t and x cannot be integers at the same time.

4. Sums of Cubes

In *The Lost Notebook*, a solution of the cubic equation $\alpha_n^3 + \beta_n^3 = \gamma_n^3 + (-1)^n$ is given in terms of three sequences α_n, β_n, and γ_n, defined by the following generating functions:

$$\left\{ \begin{aligned} \sum_{n=0}^{\infty} \alpha_n x^n &= \frac{1+53x+9x^2}{1-82x-82x^2+x^3} = 1 + 135x + 11161x^2 + \ldots \\ \sum_{n=0}^{\infty} \beta_n x^n &= \frac{2-26x-12x^2}{1-82x-82x^2+x^3} = 2 + 138x + 11468x^2 + \ldots \\ \sum_{n=0}^{\infty} \gamma_n x^n &= \frac{2+8x-10x^2}{1-82x-82x^2+x^3} = 2 + 172x + 14258x^2 + \ldots\,. \end{aligned} \right. \tag{11}$$

After verifying the result, Hirschorn offers a hypothesis on Ramanujan's possible argument [12]. Perhaps Hirschorn is right, and from his techniques a second set of solutions can easily be found using the equivalence notion of quadratic forms

[10, 7]. Hirschorn credited Ramanujan with the introduction of four quadratic forms

$$(12)\qquad \begin{cases} Q_1(A,B) = A^2 + 7AB - 9B^2 \\ Q_2(A,B) = A^2 - 9AB - B^2 \\ Q_3(A,B) = 2(A^2 + 5B^2) \\ Q_4(A,B) = 2(A^2 - 2AB + 6B^2) \end{cases}$$

satisfying the identity

$$(13)\qquad Q_1(A,B)^3 + Q_4(A,B)^3 = Q_3(A,B)^3 + Q_2(A,B)^3 .$$

Since $Q_1(A,B)$ and $Q_2(A,B)$ have the same discriminant 85, then a linear substitution

$$\begin{cases} \hat{A} = A + 8B \\ \hat{B} = B \end{cases}$$

exists which transforms $Q_1(\hat{A},\hat{B})$ into $Q_2(A,B)$. Applying the same transformation to the four quadratic forms (12), four different forms are obtained

$$\begin{cases} Q_1(A,B) \to A^2 - 9AB - B^2 \\ Q_2(A,B) \to A^2 - 25AB + 135B^2 \\ Q_3(A,B) \to 2A^2 - 32AB + 138B^2 \\ Q_4(A,B) \to 2A^2 - 36AB + 172B^2 \end{cases}$$

which still satisfy equation (13). Setting $A = h_{n+1} - 8h_n$ and $B = h_n$ in (13) and performing the same calculations that Hirschorn did, we obtain a second set of solutions

$$(14)\qquad \begin{cases} \sum_{n=0}^{\infty} \zeta_n x^n = \frac{-1+91x+135x^2}{1-82x-82x^2+x^3} = -1 + 9x + 791x^2 + \dots \\ \sum_{n=0}^{\infty} \eta_n x^n = \frac{2-154x-172x^2}{1-82x-82x^2+x^3} = 2 + 10x + 812x^2 + \dots \\ \sum_{n=0}^{\infty} \chi_n x^n = \frac{2-152x-138x^2}{1-82x-82x^2+x^3} = 2 + 12x + 1010x^2 + \dots , \end{cases}$$

where $\zeta_n^3 + \eta_n^3 = \chi_n^3 - (-1)^n$.

5. Conclusion

The beauty of Ramanujan's results makes us wonder if there are any more of his undiscovered theories. In this paper the problems have been tackled using elementary methods, making the appearance of certain solutions a seemingly casual event. Perhaps Ramanujan had some profound and mysterious theories driving his extraordinary work and, according to Berndt, "*still remain hidden behind an opaque curtain*" [5].

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On Rapidly Convergent Series for Dirichlet L-function Values Via the Modular Relation

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Dedicated to Professor R.P. Bambah on his 75th birthday

Special values of various zeta- and L-functions in number theory and related areas at integral (or almost integral) arguments have been the major subject of research over the years.

It was known to L. Euler that the value of the Riemann zeta-function $\zeta(s)$ at its positive even integral argument $s = 2n$ is a rational multiple (Bernoulli numbers come into play) of π^{2n}:

$$\zeta(2n) = \frac{(-1)^{n+1}2^{2n}}{2(2n)!}B_{2n}\pi^{2n}.$$

A similar situation prevails for the Dirichlet L-function $L(s, \chi)$ at integral arguments $s = n$ when the parity of n and χ are the same, i.e. when n is even and χ is even ($\chi(-1) = 1$) and n is odd and χ is odd ($\chi(-1) = -1$) by the work of H.L. Leopoldt (generalized Bernoulli numbers due to him show up here).

However, very little is known about arithmetic nature of $\zeta(2n + 1)$ (save for R. Apéry's famous irrationality result on $\zeta(3)$ and the most recent result due to T. Rivoal [30]), all the more so the values of $L(n, \chi)$ when n and χ have a different parity.

In this context, S. Ramanujan's very famous formula connecting the value of $\zeta(2n+1)$ to the rapidly convergent Lambert series is noteworthy and has been the resources of various kinds of generalizations, which reads

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$$\alpha^{1-n}\left\{\frac{1}{2}\zeta(2n-1)+\sum_{k=1}^{\infty}\sigma_{1-2n}(k)e^{-2k\alpha}\right\}$$
$$=(-\beta)^{1-n}\left\{\frac{1}{2}\zeta(2n-1)+\sum_{k=1}^{\infty}\sigma_{1-2n}(k)e^{-2k\beta}\right\}$$
$$-2^{2(n-1)}\sum_{r=0}^{n}(-1)^r\frac{B_{2r}}{(2r)!}\frac{B_{2n-2r}}{(2n-2r)!}\alpha^{n-r}\beta^r, \tag{1}$$

where $\alpha > 0$, $\beta > 0$ are subject to $\alpha\beta = \pi^2$ and $\sigma_a(n) = \sum_{d|n} d^a$.

For the connection with the Eichler integrals, confer C. Nagasaka [29].

There are three different approaches due to B.C. Berndt, K. Katayama, (E. Grosswald, Y. Matsuoka), and M. Toyoizumi, respectively which lead to Ramanujan's formulas for Dirichlet L-functions.

The first, due to Berndt [1], [2], [3] resorts to the transformation formula for a class of generalized Eisenstein series which contains Dedekind's η-function as a special case, under the transformation $V \in SL_2(\mathbb{Z})$, $Vz = \frac{az+b}{cz+d}$, where z lies in the upper half-plane. In the transformation formulas there arise various generalized Dedekind sums which are shown to obey reciprocity laws by the employment of the aforementioned transformation formulas.

The second approach, due to Katayama [20], [21], investigates the Lambert series directly in the spirit of C.L. Siegel and ends up with the appeal to transformation formulas for the Dedekind η-function (or generalizations thereof).

The third approach, due to Toyoizumi [33], [34] who was instigated by Matsuoka's intriguing work [26], [27], is a slight generalization of the Hecke theory [15] with a gamma factor replaced by the following alternatives (cf. in this context, Grosswald's paper [11], whose essential principle is explained in Example 4 [17]). On the one hand he considered the slightly more general functional equation than E. Hecke

$$A^{-s}\Delta(s)\varphi(s) = A^{-(\delta-s)}\Delta(\delta-s)\psi(\delta-s), \tag{2}$$

where $\varphi(s)$ and $\psi(s)$ are Dirichlet series converging in some half plane and $\Delta(s)$, denoting the gamma factor throughout in what follows, is

$$\Delta(s) = \Gamma(as), \qquad a > 0, \tag{3}$$

and from Theorem 1 [34] he deduced many formulas analogous to, and more general than, Ramanujan's formulas given theretofore.

On the other hand, he considered the same situation with

$$\Delta(s) = \Gamma(s)\Gamma\left(s+d+\frac{1}{2}\right), \tag{4}$$

where d is a non-negative integer and deduced new results for values of zeta- and L-functions at half-integral arguments, and using Theorem 2 [34] he derived all the results due to Katayama [20], Matsuoka [26], [27], and himself [33].

More specifically, Toyoizumi's Theorem 7 [34] is Matsuoka's Theorem 1 [27] regarding zeta-function values at special rational arguments, which in turn is a generalization of Ramanujan's another formula for $\zeta(1-2n)$ similar to (1), while Toyoizumi's Theorem 8 [34], which corresponds to Matsuoka's Theorem 2 [27], is concerned with the zeta-function values and contains Ramanujan's formula (1) as a special case.

We have elucidated the situation surrounding Toyoizumi's Theorem 2 [34] as Example 3 in our former paper [17] as a manifestation of the modular relation principle, postponing more detailed exposition of various consequences of Toyoizumi's Theorem 2 [34], i.e. Theorem 9, Theorem 12 (Theorem 1 of [33]), and Theorem 13 (Theorem 2 of [33]) elsewhere.

Our purpose in this paper (in §2) is to clarify the circumstances that surround Toyoizumi's Theorem 1 [34] and especially, to interpret his Theorems 10 and 11 for Dirichlet L-function values, which are the generalizations of Ramanujan's formulas for $\zeta(1-2n)$ and for $\zeta(2n-1)$, as modular relations. This gives a clear exposition of results of Berndt, Grosswald and Katayama, from a different point of view.

Other consequences of Toyoizumi's Theorem 1 [34], i.e. his Theorems 14 and 15 and his further result for $1/3$ argument [35], where [35] is a consequence of Theorem 15 which is in turn a consequence of Theorem 1, will be studied in the forthcoming paper [18], so will be the most recent result of M. Katsurada [23] on the evaluation of Lambert series of the type

$$\sum_{n=1}^{\infty} \frac{e^{-\alpha n^2 \tau}}{(1-e^{-n^2\tau})^r}$$

which includes Ramanujan's original formula as well as S. Egami's χ-analogue thereof [7]. We suspect that the latter could be viewed as the values of multiple zeta-function at $1/2$.

While Toyoizumi's theory [34] is concerned with imaginary quadratic fields ($\Delta(s) = \Gamma(s)$), Grosswald in [12] treats real quadratic fields ($\Delta(s) = \Gamma^2(s/2)$) and in [11] treats algebraic number fields in general and totally real number fields in particular, so does the paper of Egami [8].

All these will be studied in another forthcoming paper [19] of ours.

We note that the differentiated form of Guinand's type formulas (e.g.(3.6)) is more convenient for numerical computation than Ramanujan's type formulas involving Bernoulli numbers (e.g.(2.31)). In §3 of this paper we shall state an analogous formula for Dirichlet L-function and compute Catalan's constant. Other numerical computations will be given elsewhere.

In §1 we shall state rapidly convergent series expression for Dirichlet L-function which contain all the hitherto published results, notably by Katsurada [22] and H.M. Srivastava and H. Tsumura [31], in the same context as in §1 of [17]. Remarkably enough, the main source of all these is a slight generalization of another formula of Ramanujan (Berndt, Part I, Entry 28(b)).

Thus, our paper is composed of §1: rapidly convergent series, §2: Dirichlet L-function values via the modular relation, and §3: Guinand's formula.

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1. Rapidly Convergent Series Expressions for Dirichlet L-functions

Katsurada's results [22] are presented as Corollaries 1 and 2 in our paper [17], but the proof given there is a bit clumsy. Although Srivastava and Tsumura's results [31] could be read off, as even part, from our Main Theorem [17], it is not immediate.

For there two reasons, we shall give a formula which immediately gives rise to the even part [31] and the odd part [22] with a better proof.

First we fix the notation.

$$\zeta(s,\alpha) = \sum_{n=0}^{\infty} \frac{1}{(n+\alpha)^s}, \qquad \sigma = \operatorname{Re} s > 1,\ \alpha > 0$$

denotes the Hurwitz zeta-function,

$$f_\alpha(s) = \sum_{n=1}^{\infty} \frac{\sin \pi(\frac{s}{2} + 2n\alpha)}{n^{1-s}}, \qquad \sigma < 0,\ 0 < \alpha \le 1$$

is the ramified mirror image of $\zeta(s,\alpha)$,

$$\psi(s) = \frac{\Gamma'}{\Gamma}(s)$$

denotes the digamma function, and

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \qquad \sigma = \operatorname{Re} s > 1$$

denotes the Dirichlet L-function, where χ denotes a Dirichlet character to some fixed modulus k. $\mathfrak{a} = \mathfrak{a}(\chi) = \frac{1}{2}(1 - \chi(-1))$ which is 0 or 1 according as χ is even or odd.

In what follows we shall always assume χ is primitive and non-principal.

It is well-known that the L-function satisfies the functional equation

$$\begin{aligned} &\left(\frac{\pi}{k}\right)^{-(s+\mathfrak{a})/2} \Gamma\left(\frac{s+\mathfrak{a}}{2}\right) L(s,\chi) \\ &= W(\chi)\left(\frac{\pi}{k}\right)^{-(1-s+\mathfrak{a})/2} \Gamma\left(\frac{1-s+\mathfrak{a}}{2}\right) L(1-s,\overline{\chi}), \end{aligned} \tag{1.1}$$

where

$$W(\chi) = \frac{T(\chi)}{i^{\mathfrak{a}} k^{1/2}}, \tag{1.2}$$

and where $T(\chi)$ denotes the normalized Gauss sum

$$T(\chi) = \sum_{a \bmod k} \chi(a) e^{2\pi i a/k}. \tag{1.3}$$

We state a special case of Theorem A [16], which is essentially the Taylor expansion in the second variable of the derivatives of Hurwitz zeta-function (cf. [4]).

Lemma 1. *Let $k \geq 2$. Then for integers $\lambda \geq 0$ and r, $0 \leq r < k$, and a parameter x with $|x| < 1$*

$$\begin{aligned}
&\sum_{m=2}^{\infty} (-1)^m \frac{(m-1)!}{(m+\lambda)!} \zeta\left(m, \frac{r}{k}\right)\left(\frac{x}{k}\right)^m \\
&= \frac{1}{\lambda!}\left(\frac{x}{k}\right)^{-\lambda} \zeta'\left(-\lambda, \frac{r+x}{k}\right) - \frac{1}{\lambda!} \sum_{m=0}^{\lambda} \binom{\lambda}{m} \zeta'\left(-m, \frac{r}{k}\right)\left(\frac{x}{k}\right)^{-m} \\
&+ \frac{1}{\lambda!} \sum_{m=0}^{\lambda-1} \binom{\lambda}{m} \{\psi(\lambda+1) - \psi(m+1)\} \zeta\left(-m, \frac{r}{k}\right)\left(\frac{x}{k}\right)^{-m} \\
&- \frac{1}{(\lambda+1)!}\left\{\psi(\lambda+1) + \psi\left(\frac{r}{k}\right) + \gamma\right\} \frac{x}{k},
\end{aligned}$$

where γ denotes Euler's constant ($\gamma = -\psi(1)$) and the derivative $\zeta'(s, \alpha)$ is one with respect to the variable s.

From Lemma 1 we deduce, using $L(s, \chi) = k^{-s} \sum_{r=1}^{k} \chi(r) \zeta(s, \frac{r}{k})$,

Theorem 1. *Let χ be a Dirichlet character to the modulus k, $k \geq 3$. Then we have for $|x| < 1$ and any integer $\lambda \geq 0$,*

$$\begin{aligned}
&\frac{T(\overline{\chi}) \pi i^{\mathfrak{a}}}{k} \sum_{m=1}^{\infty} \frac{(m-1)!}{(m+\lambda)!} L(m, \chi)(-x)^{m-\mathfrak{a}} \\
&= \left(\frac{k}{2\pi}\right)^{\lambda} x^{-\lambda-\mathfrak{a}} \left((-\psi(\lambda+1) - \log 2\pi) \sum_{l=1}^{\infty} \frac{\overline{\chi}(l)}{l^{\lambda+1}} \sin\left(\frac{2lx}{k} - \frac{\lambda+\mathfrak{a}}{2}\right)\pi \right. \\
&\left. + \sum_{l=1}^{\infty} \frac{\overline{\chi}(l) \log l}{l^{\lambda+1}} \sin\left(\frac{2lx}{k} - \frac{\lambda+\mathfrak{a}}{2}\right)\pi + \frac{\pi}{2} \sum_{l=1}^{\infty} \frac{\overline{\chi}(l)}{l^{\lambda+1}} \cos\left(\frac{2lx}{k} - \frac{\lambda+\mathfrak{a}}{2}\right)\pi \right)
\end{aligned}$$

$$-\sum_{m=0}^{\lambda}\frac{1}{(\lambda-m)!}\left(\frac{k}{2\pi}\right)^{m}x^{-m-\mathfrak{a}}\left[\sin\left(\frac{m+\mathfrak{a}}{2}\pi\right)\{(\psi(\lambda+1)-\log 2\pi)\right.$$

$$\left.\times L(m+1,\overline{\chi})+L'(m+1,\overline{\chi})\}+\frac{\pi}{2}\cos\left(\frac{m+\mathfrak{a}}{2}\pi\right)L(m+1,\overline{\chi})\right],$$

where for $x = 0$, we understand that the formula is multiplied by $x^{\lambda+\mathfrak{a}}$ throughout.

Now, adding the above formula with x and $-x$, we obtain

Corollary 1. *The notation being the same as in Theorem* 1, *we have*

$$\frac{2T(\overline{\chi})i^{\mathfrak{a}}}{k}\sum_{j=1-\mathfrak{a}}^{\infty}\frac{(2j+\mathfrak{a}-1)!}{(2j+\lambda+\mathfrak{a})!}L(2j+\mathfrak{a},\chi)x^{2j+\mathfrak{a}}$$

$$=\left(\frac{k}{2\pi x}\right)^{\lambda}\sum_{l=1}^{\infty}\frac{\overline{\chi}(l)}{l^{\lambda+1}}\cos\left(\frac{2lx}{k}-\frac{\lambda+\mathfrak{a}}{2}\right)\pi$$

$$-\sum_{j=\mathfrak{a}}^{[(\lambda+\mathfrak{a})/2]}\frac{(-1)^{j}}{(\lambda+\mathfrak{a}-2j)!}\left(\frac{k}{2\pi x}\right)^{2j-\mathfrak{a}}L(2j-\mathfrak{a}+1,\overline{\chi}).$$

From Corollary 1 we immediately deduce

Corollary 2. *We have the even part* $(\lambda+\mathfrak{a}=2n)$

$$\frac{2T(\overline{\chi})i^{\mathfrak{a}}}{k}\sum_{j=1-\mathfrak{a}}^{\infty}\frac{(2j+\mathfrak{a}-1)!}{(2j+2n)!}L(2j+\mathfrak{a},\chi)x^{2j+\mathfrak{a}}$$

$$=\left(\frac{k}{2\pi x}\right)^{2n-\mathfrak{a}}(-1)^{n}\sum_{l=1}^{\infty}\frac{\overline{\chi}(l)}{l^{2n+1-\mathfrak{a}}}\cos\frac{2\pi lx}{k}$$

$$-\sum_{j=\mathfrak{a}}^{n}\frac{(-1)^{j}}{(2n-2j)!}\left(\frac{k}{2\pi x}\right)^{2j-\mathfrak{a}}L(2j-\mathfrak{a}+1,\overline{\chi}) \tag{1.4}$$

and the odd part $(\lambda+\mathfrak{a}=2n+1)$

$$\frac{2T(\overline{\chi})i^{\mathfrak{a}}}{k}\sum_{j=1-\mathfrak{a}}^{\infty}\frac{(2j+\mathfrak{a}-1)!}{(2j+2n+1)!}L(2j+\mathfrak{a},\chi)x^{2j+\mathfrak{a}}$$

$$=\left(\frac{k}{2\pi x}\right)^{2n-\mathfrak{a}+1}(-1)^{n}\sum_{l=1}^{\infty}\frac{\overline{\chi}(l)}{l^{2n-\mathfrak{a}+2}}\sin\frac{2\pi lx}{k}$$

$$-\sum_{j=\mathfrak{a}}^{n}\frac{(-1)^{j}}{(2n-2j+1)!}\left(\frac{k}{2\pi x}\right)^{2j-\mathfrak{a}}L(2j-\mathfrak{a}+1,\overline{\chi}). \tag{1.5}$$

From Corollary 2 we can immediately deduce

Corollary 3. *For χ even*

$$L(2n+1,\chi)-\sum_{l=1}^{\infty}\frac{\chi(l)}{l^{2n+1}}\cos\frac{2\pi lx}{k}$$

$$=(-1)^{n+1}\left(\frac{2\pi x}{k}\right)^{2n}\left\{\sum_{j=0}^{n-1}\frac{(-1)^j}{(2n-2j)!}\frac{L(2j+1,\chi)}{(2\pi x/k)^{2j}}\right.$$

$$\left.+\frac{2T(\chi)}{k}\sum_{j=1}^{\infty}\frac{(2j-1)!}{(2n+2j)!}L(2j,\overline{\chi})x^{2j}\right\}, \tag{1.6}$$

for χ odd

$$L(2n,\chi)-\sum_{l=1}^{\infty}\frac{\chi(l)}{l^{2n}}\cos\frac{2\pi lx}{k}$$

$$=(-1)^{n}\left(\frac{2\pi x}{k}\right)^{2n-1}\left\{\sum_{j=1}^{n-1}\frac{(-1)^{j-1}}{(2n-2j)!}\frac{L(2j,\chi)}{(2\pi x/k)^{2j-1}}\right.$$

$$\left.+\frac{2T(\chi)i}{k}\sum_{j=0}^{\infty}\frac{(2j)!}{(2n+2j)!}L(2j+1,\overline{\chi})x^{2j+1}\right\}, \tag{1.7}$$

for χ even

$$L(2n+1,\chi)-\frac{k}{2\pi x}\sum_{l=1}^{\infty}\frac{\chi(l)}{l^{2n+2}}\sin\frac{2\pi lx}{k}$$

$$=(-1)^{n+1}\left(\frac{2\pi x}{k}\right)^{2n}\left\{\sum_{j=0}^{n-1}\frac{(-1)^j}{(2n-2j+1)!}\frac{L(2j+1,\chi)}{(2\pi x/k)^{2j}}\right.$$

$$\left.+\frac{2T(\chi)}{k}\sum_{j=1}^{\infty}\frac{(2j-1)!}{(2n+2j+1)!}L(2j,\overline{\chi})x^{2j}\right\}, \tag{1.8}$$

and for χ odd

$$\begin{aligned} L(2n,\chi) &- \frac{k}{2\pi x}\sum_{l=1}^{\infty}\frac{\chi(l)}{l^{2n}}\sin\frac{2\pi l x}{k} \\ &= (-1)^{n+1}\left(\frac{2\pi x}{k}\right)^{2n-1}\left\{\sum_{j=1}^{n-1}\frac{(-1)^j}{(2n-2j+1)!}\frac{L(2j,\chi)}{(2\pi x/k)^{2j-1}}\right. \\ &\quad \left. - \frac{2T(\chi)i}{k}\sum_{j=0}^{\infty}\frac{(2j)!}{(2n+2j+1)!}L(2j+1,\bar{\chi})x^{2j+1}\right\}. \end{aligned} \tag{1.9}$$

Indeed, (1.6) and (1.7) are special cases of (1.4) with $a = 0$, $\lambda = 2n$ and $a = 1$, $\lambda = 2n - 1$, respectively, and (1.8) and (1.9) are special cases of (1.5) with $a = 0$, $\lambda = 2n + 1$ and $a = 1$, $\lambda = 2n$, respectively.

Remark 1. We remark that (1. 7) is Katsurada's Theorem 3, (ii) [22] (which is given as Corollary 2 in [16]) and (1.6) is the proper form of his Theorem 3, (i). The original form of Katsurada's Theorem 3, (i) (presented as Corollary 1 in [16]) is a consequence of both (1.4) and (1.5), i.e. subtracting (1.5) with $a = 0$, $\lambda = 2n - 1$ from $2n$ times (1.4) with $a = 0$, $\lambda = 2n$ yields

$$\begin{aligned} nL(2n+1,\chi) &- n\sum_{l=1}^{\infty}\frac{\chi(l)\cos(2\pi l x/q)}{l^{2n+1}} - \frac{\pi x}{q}\sum_{l=1}^{\infty}\frac{\chi(l)\sin(2\pi l x/q)}{l^{2n}} \\ &= (-1)^n\left(\frac{2\pi}{q}\right)^{2n}\left\{\sum_{k=1}^{n-1}(-1)^{k-1}\frac{kL(2k+1,\chi)}{(2n-2k)!(2\pi/q)^{2k}}x^{2n-2k}\right. \\ &\quad \left. + \frac{\tau(\chi)}{q}\sum_{k=1}^{\infty}\frac{(2k)!L(2k,\bar{\chi})}{(2n+2k)!}x^{2n+2k}\right\}, \end{aligned} \tag{1.6$'$}$$

which is a generalization of several preceding results.

Formulas (1.8) and (1.9) are stated as new results in [31] as Formulas (3. 3) and (3.4), respectively.

It is, however, obvious that all these four formulas stand on equal footing and are mere avatars of (a slight generalization of) Ramanujan's formula (Entry 28 (b), [5]).

2. The Results of Berndt, Grosswald, Katayama and Toyoizumi

As stated in Introduction, we shall render Toyoizumi's Theorem 1 [34] more visual by interpreting it (or a generalization of it) as a modular relation, and deduce other formulas from it.

First we give the definition of the functional equation studied extensively by Bochner, Chandrasekharan and Narasimhan, Berndt et al., and then recall the modular relation principle [17].

Definition 1. (i) The two Dirichlet series $\varphi(s)$ and $\psi(s)$ formed from $\{\lambda_n\}$, $\{\mu_n\}$, $\{a_n\}$, $\{b_n\}$, where

$$0 < \lambda_1 < \lambda_2 < \cdots \to \infty, \qquad 0 < \mu_1 < \mu_2 < \cdots \to \infty,$$
$$\varphi(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}, \qquad \psi(s) = \sum_{n=1}^{\infty} \frac{b_n}{\mu_n^s},$$

absolutely convergent in some half-plane, are said to satisfy the functional equation with multiple gamma factor $\Delta(s)$ and constants $A > 0$, $\delta, c \in \mathbb{C}$:

$$A^{-s}\Delta(s)\varphi(s) = cA^{-(\delta-s)}\Delta(\delta - s)\psi(\delta - s) \tag{2.1}$$

if there exists a function $\chi(s) = \chi_A(s)$ holomorphic outside a compact set $\mathcal{S}$, convex in any finite strip coinciding with $A^{-s}\Delta(s)\varphi(s)$ and $cA^{-(\delta-s)}\Delta(\delta - s)$ $\psi(\delta - s)$ in the respective region of absolute convergence.

(ii) We also define the residual function

$$\begin{aligned} P(x) = P_A(x) &= \frac{1}{2\pi i}\int_C \chi(s)x^{-s}ds \\ &= \frac{1}{2\pi i}\int_C A^{-s}\Delta(s)\varphi(s)x^{-s}ds, \end{aligned} \tag{2.2}$$

where C encloses all of $\mathcal{S}$ (thus $P(x)$ is in most cases the sum of the residues of the integrand in the strip $\operatorname{Re}\delta - \kappa < \sigma < \kappa$, where κ is greater than the abscissa of absolute convergence of $\varphi(s)$).

Supposing that $\Delta(s)$ is holomorphic in $\sigma > 0$ and $\Delta(\sigma + it)$ belongs to $L_1(\mathbb{R})$ uniformly in $0 < \sigma_1 \leq \sigma \leq \sigma_2$, we set up the inverse Mellin integral ($\kappa \gg 1$ and $\operatorname{Re} z > 0$)

$$E(z) = \frac{1}{2\pi i}\int_{(\kappa)} \Delta(s)z^{-s}ds, \tag{2.3}$$

where $\int_{(\kappa)}$ denotes the vertical integral $\int_{\kappa-i\infty}^{\kappa+i\infty}$.

Putting

$$\Phi(z) = \sum_{n=1}^{\infty} a_n E(\lambda_n z), \quad \Psi(z) = \sum_{n=1}^{\infty} b_n E(\mu_n z), \tag{2.4}$$

we form the Mellin transforms

$$\chi_1(s) = \chi_{1,A}(s) = \int_0^\infty \left(\frac{x}{A}\right)^s \Phi(x)\frac{dx}{x},$$

$$\chi_2(s) = \chi_{2,A}(s) = \int_0^\infty \left(\frac{x}{A}\right)^s \Psi(x)\frac{dx}{x}. \tag{2.5}$$

Then the functional equation (2.1) takes the form

$$\chi_1(s) = c\chi_2(\delta - s). \tag{2.6}$$

The main ingredient in what follows is the following.

The modular relation principle. Under the assumption that the Mellin integrals (2.5) are absolutely convergent in a right half-plane, and that $E(\xi + i\eta) = O(e^{-c_1\xi}),\quad \xi \to \infty$ for some $c_1 > 0$, the functional equation (2.6) implies the modular relation

$$\Phi(z) = c\left(\frac{A}{z}\right)^\delta \Psi\left(\frac{A^2}{z}\right) + P_A\left(\frac{z}{A}\right), \tag{2.7}$$

or

$$\Phi(Az) = cz^{-\delta}\Psi\left(\frac{A}{z}\right) + P_A(z), \tag{2.7$'$}$$

where

$$P_A(z) = \frac{1}{2\pi i}\int_C \Delta(s)\varphi(s)(Az)^{-s}ds, \tag{2.8}$$

C being as in Definition 1.

Conversely, (2.7) implies (2.6), and *a fortiori* (2.1).

Some authors refer to this relation as the Riemann-Hecke-Bochner correspondence [24].

Theorem 2. *Toyoizumi's Theorem* 1 [34] *is the modular relation, i.e. his functional equation*

$$A^{-s}\Delta(s)\varphi(s) = cA^{-(\delta-s)}\Delta(\delta - s)\psi(\delta - s), \tag{2.9}$$

where

$$\Delta(s) = \Gamma(as), \quad a > 0, \tag{2.10}$$

and

$$\delta = \frac{b}{a} \in \mathbb{R}, \quad A = \left(\frac{2a\pi}{\lambda}\right)^a > 0, \quad \lambda = \sqrt{k} > 0,$$

is equivalent to the modular relation

$$\Phi(A\tau^a) = c\tau^{-a\delta}\Psi(A\tau^{-a}) + P_A(\tau^a), \tag{2.11}$$

or in the form of a series, to

$$\tilde{f}(\pi\tau) = c\tau^{-a\delta}\tilde{g}(\pi\tau^{-1}) + aP_A(\tau^a), \tag{2.12}$$

where

$$\begin{aligned} \tilde{f}(\pi\tau) &= a\Phi(A\tau^a) = \sum_{n=1}^{\infty} a_n e^{-(An)^{1/a}\tau} \\ &= \sum_{n=1}^{\infty} a_n e^{-\frac{2a}{\lambda}\sqrt[a]{n}\pi\tau}, \end{aligned} \tag{2.13}$$

and

$$\tilde{g}(\pi\tau) = a\Psi(A\tau^a)$$

is the same as $\tilde{f}(\pi\tau)$ *with* a_n *replaced by* b_n.

Further, putting

$$\tilde{F}(x) = x^{b/2}\tilde{f}(x), \quad \tilde{G}(x) = x^{b/2}\tilde{g}(x), \tag{2.14}$$

where b *is the numerator of the abscissa of the central line* $\delta = \frac{b}{a}$, *then* (2.9) *is equivalent to*

$$\tilde{F}(\alpha) = c\tilde{G}(\beta) + a\alpha^{b/2}P_A\left(\left(\frac{\alpha}{\pi}\right)^a\right) \tag{2.15}$$

with $\alpha = \pi\tau$, $\beta = \pi\tau^{-1}$ *giving* $\alpha\beta = \pi^2$.

Proof. Since $\Delta(s) = \Gamma(as)$, the inverse Mellin transform (2.3) reduces to

$$E(z) = \frac{1}{a}\frac{a}{2\pi i}\int_{(\kappa)} \Gamma(as)(z^{1/a})^{-as}ds = \frac{1}{a}e^{-\sqrt[a]{z}} \qquad (\operatorname{Re} z > 0)$$

directly or as in Toyoizumi's Lemma 1 [34].

Therefore, (2.7)′ reads

$$\frac{1}{a}\sum_{n=1}^{\infty} a_n e^{-(A\tau^a n)^{1/a}} = \frac{c}{a}\tau^{-a\delta}\sum_{n=1}^{\infty} b_n e^{-(A\tau^{-a} n)^{1/a}} + P_A(\tau^a), \tag{2.16}$$

or (2.12), which coincides with the assertion of Lemma 3 [34]:

$$\tilde{f}(\pi\tau^{-1}) = c\tau^b \tilde{g}(\pi\tau) + aP_A(\tau^{-a}).$$

Multiplying both sides of (2.12) by $(\pi\tau)^{b/2}$ and putting $\alpha = \pi\tau$, $\beta = \pi\tau^{-1}$ gives rise to (2.15). This completes the proof. □

Remerk 2. Considering the sequence $\{n^\alpha\}$ in place of $\{n\}$ was already suggested by G.H. Hardy [14] who had been motivated by Hj. Mellin's work [28] (cf. also A.O. Gel'fond's exposition [9]).

Thus, Toyoizumi's Theorem 1 [34] may be viewed as partial substantiation of Hardy's enunciation.

In another, but similar, context, Hardy's anticipation was manifested by A.F. Lavrik [25].

In this paper, however, Theorem 2 would be used with $a = 1$ only, and in [18] it will be used in its full strength.

To study the modular relation for Dirichlet L-function, we introduce the notation.

Hereafter we assume χ be a non-principal primitive Dirichlet character to the modulus k and we let $L(s, \chi)$ be the associated Dirichlet L-function defined in §1.

For any $z \in \mathbb{C}$ we introduce the arithmetic functions following Grosswald [10] and Toyoizumi [34]:

$$\sigma_{z,\chi}(n) = \sum_{d|n} \chi(d) d^z \tag{2.17}$$

and

$$\sigma^*_{z,\chi}(n) = \sum_{d|n} \chi\left(\frac{n}{d}\right) d^z, \tag{2.18}$$

where the summation extends over all positive divisors d of n.

These appear as coefficients of Dirichlet series for the following products of the Riemann zeta- and Dirichlet L-function, respectively:

$$\varphi(s) = \zeta(s)L(s+z, \chi) = \sum_{n=1}^{\infty} \sigma_{-z,\chi}(n) n^{-s} \tag{2.19}$$

and

$$\psi(s) = \zeta(s+z)L(s,\overline{\chi}) = \sum_{n=1}^{\infty} \sigma^{*}_{-z,\overline{\chi}}(n)n^{-s}. \tag{2.20}$$

We note the relation

$$\sigma_{z,\chi}(n) = n^{z}\sigma^{*}_{-z,\chi}(n).$$

Now from the well-known functional equations for the Riemann zeta-function and Dirichlet L-functions (1.1), it follows that

$$\begin{aligned} &\pi^{-s}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+z+\mathfrak{a}}{2}\right)\varphi(s) \\ &= W(\chi)k^{\frac{1}{2}-z-s}\pi^{-(1-z-s)} \\ &\times\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{1-z-s+\mathfrak{a}}{2}\right)\psi(1-z-s), \end{aligned} \tag{2.21}$$

where $\mathfrak{a}$, $W(\chi)$ and $T(\chi)$ are as defined in §1.

We shall treat only the case

$$z+\mathfrak{a} = 2m-1, \quad m \in \mathbb{N}, \tag{2.22}$$

in which case the gamma factor appearing in (2.21) reduces, by the use of the functional equation and the duplication formula for the gamma function, to a simple $\Gamma(s)$:

$$\begin{aligned} \left(\frac{2\pi}{\sqrt{k}}\right)^{-s}\Gamma(s)\varphi(s) &= W(\chi)k^{-\frac{z}{2}}(-1)^{\frac{z-1+\mathfrak{a}}{2}}\left(\frac{2\pi}{\sqrt{k}}\right)^{-(1-z-s)} \\ &\quad\times\Gamma(1-z-s)\psi(1-z-s). \end{aligned} \tag{2.23}$$

This fits the functional equation (2.1) with

$$A = \frac{2\pi}{\sqrt{k}}, \quad c_A(\chi) = W(\chi)k^{-\frac{z}{2}}(-1)^{\frac{z-1+\mathfrak{a}}{2}}, \quad \delta = 1-z,$$

and so the modular relation principle applies to yield

Theorem 3. *For* $\operatorname{Re}\tau > 0$ *and* $z = 2m-1-\mathfrak{a}$, *we have the modular relation*

$$\sum_{n=1}^{\infty}\sigma_{-z,\chi}(n)e^{-An\tau} = c_A(\chi)\tau^{z-1}\sum_{n=1}^{\infty}\sigma^{*}_{-z,\overline{\chi}}(n)e^{-An\tau^{-1}} + P_A(\tau), \tag{2.24}$$

where

$$c = c_A(\chi) = W(\chi)k^{-\frac{z}{2}}(-1)^{\frac{z-1+\mathfrak{a}}{2}}$$

and

$$P_A(\tau) = -\frac{1}{2}L(z,\chi) + \frac{1}{A\tau}L(2m-\mathfrak{a},\chi)$$

$$-\sum_{j=1}^{m-1}\frac{(A\tau)^{2j-1}}{(2j-1)!}\zeta(1-2j)L(2m-2j-\mathfrak{a},\chi). \tag{2.25}$$

Indeed, we have only to calculate the residues at $s = 1, 0, -1, -3, \dots, -2(m-\mathfrak{a})+1$ of the integrand $(A\tau)^{-s}\Gamma(s)\zeta(s)L(s+z,\chi)$, using the fact that $\mathrm{Res}_{s=-n}\Gamma(s) = \frac{(-1)^n}{n!}$, $n \geq 0$.

Corollary 4. (Berndt [2]-Katayama [21])

$$\frac{1}{2}L(2m-1-\mathfrak{a},\chi) + F_1(2m-1-\mathfrak{a},x,\chi)$$

$$-(ix)^{2m-2-\mathfrak{a}}T(\overline{\chi})^{-1}F_2(2m-1-\mathfrak{a},x^{-1},\chi)$$

$$= \pi^{-1}\sum_{j=0}^{m-1}(-1)^{j-1}x^{2j-1}\zeta(2j)L(2m-2j-\mathfrak{a},\chi), \tag{2.26}$$

where

$$F_1(z,x,\chi) = \sum_{n=1}^{\infty}\frac{\chi(n)}{n^z}\frac{1}{e^{2\pi nx}-1}, \tag{2.27}$$

$$F_2(z,x,\chi) = \sum_{h=1}^{k-1}\overline{\chi}(h)\sum_{n=1}^{\infty}\frac{1}{n^z}\frac{e^{2\pi nxh/k}}{e^{2\pi nx}-1}. \tag{2.28}$$

Proof. We rewrite (2.24), by putting $\frac{\tau}{\sqrt{k}} = x$ and using $T(\chi)T(\overline{\chi}) = \chi(-1)k$, as

$$\frac{1}{2}L(2m-1-\mathfrak{a},\chi) + \sum_{n=1}^{\infty}\sigma_{1+\mathfrak{a}-2m,\chi}(n)e^{-2\pi nx}$$

$$-\chi(-1)(ix)^{2m-2-\mathfrak{a}}T(\overline{\chi})^{-1}\sum_{n=1}^{\infty}\sigma^*_{1+\mathfrak{a}-2m,\overline{\chi}}(n)e^{-2\pi nx^{-1}/k}$$

$$= -\pi^{-1}\sum_{j=0}^{m-1}(-1)^{j}x^{2j-1}\zeta(2j)L(2m-2j-\mathfrak{a},\chi), \tag{2.29}$$

where in the transformation of $P_A(\tau)$ we have used the functional equation

$$\zeta(2n) = \frac{(-1)^n(2\pi)^{2n}}{2(2n-1)!}\zeta(1-2n).$$

We shall now show that all but one term ($\frac{1}{2}L(2m-1-\mathfrak{a},\chi)$) in (2.26) can be expressed in terms of Leopoldt's generalized Bernoulli numbers, $B_{n,\chi}$, which are defined by

$$\sum_{a=1}^{k} \frac{\chi(a)te^{at}}{e^{kt}-1} = \sum_{n=1}^{\infty} B_{n,\chi}\frac{t^n}{n!}. \tag{2.30}$$

Corollary 5. *In the same notation of Corollary 4, we have*

$$\begin{aligned} &\frac{1}{2}L(2m-1-\mathfrak{a},\chi)+F_1(2m-1-\mathfrak{a},x,\chi)\\ &\quad -(ix)^{2m-2-\mathfrak{a}}T(\overline{\chi})^{-1}F_2(2m-1-\mathfrak{a},x^{-1},\chi)\\ &\quad =\sum_{j=0}^{m-1}\frac{(2\pi x)^{2j-1}}{(2j)!}\,\frac{(-1)^{m-j}}{2i^{\mathfrak{a}}}\,\frac{T(\chi)(2\pi/k)^{2m-2j-\mathfrak{a}}}{(2m-2j-\mathfrak{a})!}B_{2j}B_{2m-2j-\mathfrak{a},\overline{\chi}}. \end{aligned} \tag{2.31}$$

Proof. Proof follows from the expressions

$$\zeta(1-2n)=-\frac{B_{2n}}{2n}$$

and

$$L(n,\chi)=(-1)^{\frac{n-\mathfrak{a}}{2}+1}\frac{T(\chi)}{2i^{\mathfrak{a}}}\left(\frac{2\pi}{k}\right)^n\frac{B_{n,\overline{\chi}}}{n!}, \tag{2.32}$$

which is in turn a consequence of the expression $L(1-n,\chi)=-\frac{B_{n,\chi}}{n}$ $(n\in\mathbb{N})$ and the functional equation (1.1). Here the vanishing of $B_{n,\chi}$ for $n\not\equiv\mathfrak{a}\pmod 2$ corresponds to that of B_{2j+1} or $\zeta(-2j)$ for $j\ge 1$.

Corollary 6. (Toyoizumi [34], Theorems 10 and 11) *In the notation of Theorem 2 we have*

$$\begin{aligned} \tilde{F}(\alpha)+c\tilde{G}(\alpha) &= \tilde{F}(\beta)+c\tilde{G}(\beta)+\alpha^{1-m+\frac{\mathfrak{a}}{2}}P_A\left(\frac{\alpha}{\pi}\right)\\ &\quad -\beta^{1-m+\frac{\mathfrak{a}}{2}}P_A\left(\frac{\beta}{\pi}\right) \end{aligned} \tag{2.33}$$

or in the form of series,

$$\alpha^{1-m+\frac{\mathfrak{a}}{2}}\left\{\sum_{n=1}^{\infty}(\sigma_{1-2m+\mathfrak{a},\chi}(n)+c\sigma^*_{1-2m+\mathfrak{a},\overline{\chi}}(n))e^{-\frac{2}{\sqrt{k}}n\alpha}\right.$$
$$\left.+\frac{1}{2}L(2m-1-\mathfrak{a},\chi)\right\}$$

$$
\begin{aligned}
&= \beta^{1-m+\frac{\mathfrak{a}}{2}}\left\{\sum_{n=1}^{\infty}(\sigma_{1-2m+\mathfrak{a},\chi}(n)+c\sigma^{*}_{1-2m+\mathfrak{a},\overline{\chi}}(n))e^{-\frac{2}{\sqrt{k}}n\beta}\right.\\
&\left.+\frac{1}{2}L(2m-1-\mathfrak{a},\chi)\right\}\\
&+\sqrt{k}\sum_{j=0}^{m-1}\frac{(-1)^{j+1}\alpha^{2j-m+\frac{\mathfrak{a}}{2}}\zeta(2j)L(2m-2j-\mathfrak{a},\chi)}{k^{j}\pi^{2j}}\\
&-\sqrt{k}\sum_{j=0}^{m-1}\frac{(-1)^{j+1}\beta^{2j-m+\frac{\mathfrak{a}}{2}}\zeta(2j)L(2m-2j-\mathfrak{a},\chi)}{k^{j}\pi^{2j}}.
\end{aligned}
$$

Proof. We apply (2.15) with $a = 1$ and $\delta = b = 1 - z = 2 - 2m + \mathfrak{a}$:

$$
\begin{aligned}
\tilde{F}(\alpha) &= c\tilde{G}(\beta) + \alpha^{1-m+\frac{\mathfrak{a}}{2}}P_A\left(\frac{\alpha}{\pi}\right)\\
\tilde{F}(\beta) &= c\tilde{G}(\alpha) + \beta^{1-m+\frac{\mathfrak{a}}{2}}P_A\left(\frac{\beta}{\pi}\right).
\end{aligned}
\tag{2.34}
$$

Formula (2.33) immediately follows from (2.34).

It remains to rewrite the residual function $P_A(x)$, which is easily done using (2.25). This completes the proof. □

Finally, we consider the results of Grosswald [10] who considered for $z = 2m - 1$,

$$
\begin{aligned}
\varphi(s) &= L(s,\chi)L(s+z,\overline{\chi})\\
&= \sum_{n=1}^{\infty}a_n(\chi)n^{-s}, \qquad \sigma > 1,
\end{aligned}
$$

which satisfies Hecke's functional equation (2.1) with $A = \frac{2\pi}{k}$, $\delta = 1 - z$, $c = c(\chi) = (-1)^{\frac{z-1}{2}+\mathfrak{a}}$, $\Delta(s) = \Gamma(s)$ and $\psi(s) = \varphi(s)$.

Hence the modular relation principle (2.7)′ gives rise to

$$
\begin{aligned}
\sum_{n=1}^{\infty}a_n(\chi)e^{-\frac{2\pi}{k}n\tau} &= (-1)^{\frac{z-1}{2}+\mathfrak{a}}\tau^{z-1}\sum_{n=1}^{\infty}a_n(\chi)e^{-\frac{2\pi}{k}n\tau^{-1}}\\
&\quad +P\left(\frac{2\pi}{k}\tau\right),
\end{aligned}
\tag{2.35}
$$

where

$$
P(w) = \frac{1}{2\pi i}\int_{C}w^{-s}\Gamma(s)L(s,\chi)L(s+z,\overline{\chi})ds
$$

is the residual function, the sum of the residues of the integrand in the strip $1 - z - \kappa < \sigma < \kappa$ (with $1 < \kappa < 2$).

Noting that the poles of the gamma function are cancelled by the zeros at $s = -2\nu + \mathfrak{a}$, $\nu \in \mathbb{N}$, of the L-function $L(s, \chi)$, we can easily calculate $P(w)$ as

$$\begin{aligned} P(w) &= L(0, \chi)L(z, \overline{\chi}) + \frac{T(\overline{\chi})}{2i^{\mathfrak{a}}}\left(\frac{2\pi}{k}\right)^{z}(-1)^{\frac{z+1}{2}} \\ &\quad \times \sum_{\nu=1}^{(z-1)/2}(-1)^{\nu-1}\left(\frac{kw}{2\pi}\right)^{2\nu-1+\mathfrak{a}}\frac{B_{2\nu+\mathfrak{a},\chi}}{(2\nu+\mathfrak{a})!}\frac{B_{z+1-\mathfrak{a}-2\nu,\chi}}{(z+1-\mathfrak{a}-2\nu)!}. \end{aligned} \tag{2.36}$$

Then

Corollary 7. (Grosswald [10]) *With*

$$a_n(\chi) = \sum_{d|n}\chi\left(\frac{n}{d}\right)d^{-z}\overline{\chi}(d) = \chi(n)\sum_{d|n}d^{-z}\overline{\chi}^2(d), \tag{2.37}$$

we have the modular relation

$$\begin{aligned} \sum_{n=1}^{\infty}a_n(\chi)e^{-\frac{2\pi}{k}n\tau} &= (-1)^{\frac{z-1}{2}+\mathfrak{a}}\tau^{z-1}\sum_{n=1}^{\infty}a_n(\chi)e^{-\frac{2\pi}{k}n\tau^{-1}} \\ &\quad + P\left(\frac{2\pi}{k}\tau\right), \end{aligned} \tag{2.38}$$

where $P(w)$ is given by (2.36).

Remark 3. In their original statement of the residual function, in each of Berndt [2], Katayama [20], Grosswald [10], and Toyoizumi [34], the range of summation looks different, but indeed, it is from 0 to $m - 1$ in all the cases (on noting that $L(0, \chi) = 0$ for χ even).

3. Guinand's Formula and a Numerical Example

We shall take up the functional equation (2.21) with general z, in which case the gamma factor $\Delta(s)$ does not necessarily degenerate to a single gamma function. Nevertheless, the Mellin integral (2.3) can be expressed in term of the modified Bessel function

$$\frac{1}{2\pi i}x^{-\frac{\mu}{2}}\int_{(\kappa)}x^{-s}\Gamma\left(s+\frac{\mu+\nu}{2}\right)\Gamma\left(s+\frac{\mu-\nu}{2}\right)ds = 2K_\nu(2\sqrt{x}) \tag{3.1}$$

([32]).

Recall that (2.21) fits (2.1) with the descriptions:

$$\lambda_n = \mu_n = n, \quad A = \frac{\pi}{\sqrt{k}}, \quad c = c(\chi) = W(\chi)k^{-\frac{z}{2}},$$

$$\delta = 1 - z, \quad \Delta(s) = \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+z+\mathfrak{a}}{2}\right),$$

and

$$\varphi(s) = \psi(s) = \zeta(s)L(s+z,\chi) = \sum_{n=1}^{\infty} \sigma_{-z,\chi}(n)n^{-s}.$$

Hence it follows from (3.1) and the symmetry property $K_{-\nu}(x) = K_\nu(x)$ that

$$E(\tau) = 4\tau^{(z+\mathfrak{a})/2}K_{(z+\mathfrak{a})/2}(2\tau), \tag{3.2}$$

and the modular relation (2.7)′ with $\tau = \sqrt{k}x$ implies

Theorem 4.

$$4(\pi x)^{\frac{z+\mathfrak{a}}{2}} \sum_{n=1}^{\infty} \sigma_{-z,\chi}(n)n^{\frac{z+\mathfrak{a}}{2}} K_{\frac{z+\mathfrak{a}}{2}}(2\pi nx)$$

$$= 4W(\chi)x^{-1+\frac{z-\mathfrak{a}}{2}}\pi^{\frac{z+\mathfrak{a}}{2}}k^{-\frac{z+1+\mathfrak{a}}{2}} \sum_{n=1}^{\infty} \sigma^*_{-z,\chi}(n)n^{\frac{z+\mathfrak{a}}{2}} K_{\frac{z+\mathfrak{a}}{2}}(2\pi nx^{-1})$$

$$+P_A(\sqrt{k}x), \tag{3.3}$$

where

$$P_A(w) = -\Gamma\left(\frac{z+\mathfrak{a}}{2}\right)L(z,\chi) + \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1+z+\mathfrak{a}}{2}\right)L(1+z,\chi)w^{-1}. \tag{3.4}$$

Remark 4. Theorem 4 is similar to, but more general than, Example 3 [17], Guinand's Theorem [13]. Berndt [2] seems to be the first who stated a general modular relation with general K-Bessel function (cf. Remark 1, (ii) [17]).

As in Example 3 [17], if $z + \mathfrak{a}$ is an odd integer, then $K_{(z+\mathfrak{a})/2}(w)$ reduces to an elementary function

$$K_{n+1/2}(w) = \left(\frac{\pi}{2w}\right)^{1/2} e^{-w} \sum_{r=0}^{n} \frac{(n+r)!}{r!\,(n-r)!\,(2w)^r}. \tag{3.5}$$

Hence we may write down (3.3) in the form of series with elementary function coefficients.

We shall give the simplest numerical example of this process, i.e. numerical data for Catalan's constant

$$L(2,\chi)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2},$$

where $\chi=\chi_4$ denotes the unique odd Dirichlet character mod 4.

Our computation depends on the following formula (whose general form similar to Theorem 4 [17] will be given elsewhere):

For $\chi=\chi_4$, we have

$$\begin{aligned} L(2,\chi) \;=\;& \frac{\pi^2}{8}-2\pi\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{2m-1}\\ &\times\left(\frac{1}{e^{\pi(2m-1)}-1}+\frac{1}{(e^{\pi(2m-1)}-1)^2}\right)\\ &-2\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{(2m-1)^2}\,\frac{1}{e^{\pi(2m-1)}-1}\\ &+\frac{\pi}{2}\sum_{m=1}^{\infty}\frac{1}{m}\cdot\left(\frac{1}{e^{\pi m}-e^{-3\pi m}}+\frac{4e^{-3\pi m}}{(e^{\pi m}-e^{-3\pi m})^2}\right)\\ &-\frac{\pi}{2}\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{3}{e^{3\pi m}-e^{-\pi m}}+\frac{4e^{-\pi m}}{(e^{3\pi m}-e^{-\pi m})^2}\right). \end{aligned}\tag{3.6}$$

To prove (3.6) we specialize Formula (3.3) to the case

$$\mathrm{a}=1,\ \ z=2\quad\text{and}\quad \chi=\chi_4,$$

in which case we may use (3.5). Then we are to express infinite series of the type

$$\sum_{n=1}^{\infty}\sigma_{-2,\chi}(n)n^{a}e^{-\pi n}$$

in terms of Lambert series, as in the proof of Theorem 4 [17], and then finally put $x=1/2$.

We note that $\pi^2/8$ comes from the value $L(3,\chi)$, which is $\pi^3/32$.

From (3.6), it is easily seen that the speed of convergence is such that after taking first $N-1$ terms, the absolute value of error is $< 1.65\,e^{-\pi N}/N$, so that for $N=72$, we have the following valid up to 100 decimal places:

$$\begin{aligned} L(2,\chi_4) = {}& 0.91596559417721901505460351493238411077414937428167\\ & 21342664981196217630197762547694793565129261151062 4 \end{aligned}$$

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On a Conjecture of Andrews-II

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The case $k = a$ of the 1974 conjecture of Andrews on two partition functions $A_{\lambda,k,a}(n)$ and $B_{\lambda,k,a}(n)$ was proved by the first author and T.G. Sudha [On a conjecture of Andrews, Internat. J. Math. and Math. Sci. Vol. 16, No. 4 (1993), 763–774]. In this paper we prove the two cases of $k = a + 1$ and $k = a + 2$ of the same conjecture.

1. Introduction

For an even integer λ, let $A_{\lambda,k,a}(n)$ denote the number of partitions of n into parts such that no part $\not\equiv 0 (mod\, \lambda + 1)$ may be repeated and no part is $\equiv 0, \pm(a - \frac{\lambda}{2})(\lambda + 1) mod\, [(2k - \lambda + 1)(\lambda + 1)]$. For an odd integer λ, let $A_{\lambda,k,a}(n)$ denote the number of partitions of n into parts such that no part not congruent to 0 $(mod\, \frac{\lambda+1}{2})$ may be repeated, no part is $\equiv \lambda + 1 (mod\, 2\lambda + 2)$ and no part is $\equiv 0, \pm(2a - \lambda)(\frac{\lambda+1}{2})\, mod\, [(2k - \lambda + 1)(\lambda + 1)]$.

Let $B_{\lambda,k,a}(n)$ denote the number of partitions of n of the form $b_1 + \cdots + b_s$ with $b_i \geq b_{i+1}$, no part $\not\equiv 0 (mod\, \lambda + 1)$ is repeated, $b_i - b_{i+k-1} \geq \lambda + 1$ with strict inequality if $\lambda + 1/b_i$ and $\sum_{i=j}^{\lambda-j+1} f_i \leq a - j$ for $1 \leq j \leq \frac{\lambda+1}{2}$ and $f_1 + \cdots + f_{\lambda+1} \leq a - 1$ where f_i is the number of appearances of j in the partition.

Andrews [1] conjectured the following identities for $A_{\lambda,k,a}(n)$ and $B_{\lambda,k,a}(n)$.

Conjecture. For $\frac{\lambda}{2} < a \leq k < \lambda$,

$$B_{\lambda,k,a}(n) = A_{\lambda,k,a}(n)$$

for $0 \leq n < \frac{(k+\lambda-a+1)(k+\lambda-a)}{2} + (k - \lambda + 1)(\lambda + 1)$, while

$$B_{\lambda,k,a}(n) = A_{\lambda,k,a}(n) + 1$$

when $n = \frac{(k+\lambda-a+1)(k+\lambda-a)}{2} + (k - \lambda + 1)(\lambda + 1)$.

For $3 \leq \lambda \leq 7$, $\frac{\lambda}{2} < k \leq min(\lambda - 1, 5)$, $\frac{\lambda}{2} < a \leq k$, this conjecture has been verified [1]. The case $k = a$ of this conjecture has been proved by the first author and T.G. Sudha [2]. The object of this paper is to prove the cases of $k = a + 1$ and $a + 2$ of this conjecture. Throughout we assume $k > a$.

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2. Proof

We prove the cases $k = a + 1$ and $a + 2$ of the conjecture by establishing the following identities. We note that n in the conjecture is equal to $(2k - a - \frac{\lambda}{2} + 1)$ $(\lambda + 1)$ when $k = a + 1$ and $(2k - a - \frac{\lambda}{2} + 1)(\lambda + 1) + 1$ when $k = a + 2$.

Case 1. Let λ be even. Then

1. $B_{\lambda,k,a}(n) = A_{\lambda,k,a}(n)$ for $n < (a - \frac{\lambda}{2})(\lambda + 1)$
2. $B_{\lambda,k,a}(n) = A_{\lambda,k,a}(n)$ for $n = (a - \frac{\lambda}{2})(\lambda + 1)$
3. $B_{\lambda,k,a}(n) = A_{\lambda,k,a}(n)$ for $(a - \frac{\lambda}{2})(\lambda + 1) < n < (2k - a - \frac{\lambda}{2} + 1)(\lambda + 1)$
4. $B_{\lambda,k,a}[(2k - a - \frac{\lambda}{2} + 1)(\lambda + 1)]$
   $= A_{\lambda,k,a}[(2k - a - \frac{\lambda}{2} + 1)(\lambda + 1)] + 1$ when $k = a + 1$
5. $B_{\lambda,k,a}[(2k - a - \frac{\lambda}{2} + 1)(\lambda + 1) + 1]$
   $= A_{\lambda,k,a}[(2k - a - \frac{\lambda}{2} + 1)(\lambda + 1) + 1] + 1$ when $k = a + 2$

We first observe that (1) and (2) have already been proved in [2]. Let $P_{B_{\lambda,k,a}}(n)$ and $P_{A_{\lambda,k,a}}(n)$ denote the set of partitions enumerated by $B_{\lambda,k,a}(n)$ and $A_{\lambda,k,a}(n)$ respectively. Let $P'_A(n)$ [resp. $P'_B(n)$] denote the set of partitions enumerated by $A_{\lambda,k,a}(n)$ [resp. $B_{\lambda,k,a}(n)$] but not by $B_{\lambda,k,a}(n)$ [resp. $A_{\lambda,k,a}(n)$].

Proof of (3). To prove (3) we establish a bijection of $P'_A(n)$ onto $P'_B(n)$ where $n = (2k - a - \frac{\lambda}{2})(\lambda + 1) + \theta$, $\theta < \lambda + 1$. If a partition belongs to $P'_A(n)$ then it violates one of the conditions on $f's$ or $b's$. Let S_j $(j = 1, 2, \ldots, \frac{\lambda}{2})$ denote the condition $\sum_{i=j}^{\lambda-j+1} f_i \le a - j$ and let S denote the condition $\sum_{i=1}^{\lambda+1} f_i \le a - 1$ and let S^* be the condition on $b's$. In the following steps 1 to $\frac{\lambda}{2} + 2$, we enumerate the partitions in P_A violating $S_{\frac{\lambda}{2}}, \ldots, S_1, S$ and S^* and also give the necessary bijection of $P'_A(n)$ onto $P'_B(n)$.

Step 1. Consider $S_{\frac{\lambda}{2}} : f_{\frac{\lambda}{2}} + f_{\frac{\lambda}{2}+1} \le 2 \le a - \frac{\lambda}{2}$. Since $f_{\frac{\lambda}{2}} + f_{\frac{\lambda}{2}+1} \le 2$ it is clear that there are no partitions in P_A violating $S_{\frac{\lambda}{2}}$ if $a - \frac{\lambda}{2} \ge 2$. Let $a - \frac{\lambda}{2} = 1$. Then $n = (2k - \lambda - 1)(\lambda + 1) + \theta < (2k - \lambda)(\lambda + 1)$. The set of partitions violating $S_{\frac{\lambda}{2}}$ is $\{\pi + (\frac{\lambda}{2} + 1) + \frac{\lambda}{2} + \pi' : \pi$ is a partition of θ_1 into parts $> (\frac{\lambda}{2} + 1)$ such that $\lambda + 1$ is not a part and $\alpha(\lambda + 1)$, $1 < \alpha < 2k - \lambda$ can be repeated and π' is a partition of θ_2 into distinct parts $< \frac{\lambda}{2}$ such that $\theta_1 + \theta_2 = n - (\lambda + 1)\}$. π and π' may not exist in some cases. Here we consider those partitions containing $\frac{\lambda}{2} + 1$ and $\frac{\lambda}{2}$ in which the number of parts $1, 2, \ldots, \lambda$ is $< a + 1$ while the other cases will be dealt in Step $\frac{\lambda}{2}$. Now the associated partition is $\pi^* \in P'_B$ which is obtained by rearranging $\pi + (\lambda + 1) + \pi'$ in the decreasing order.

Step 2. Consider $S_{\frac{\lambda}{2}-1} : f_{\frac{\lambda}{2}-1} + f_{\frac{\lambda}{2}} + f_{\frac{\lambda}{2}+1} + f_{\frac{\lambda}{2}+2} \le a - \frac{\lambda}{2} + 1$. Since the left hand side sum of this inequality is ≤ 4 we see that a partition violates $S_{\frac{\lambda}{2}-1}$ only if $a - \frac{\lambda}{2} + 1 < 4$ or $a - \frac{\lambda}{2} < 3$. Thus for $a - \frac{\lambda}{2} \ge 3$ there are no partitions in P_A violating $S_{\frac{\lambda}{2}-1}$. Let $a - \frac{\lambda}{2} = 1$. Then $(\lambda + 1) < n < (2k - \lambda)(\lambda + 1)$. The set of partitions violating $S_{\frac{\lambda}{2}-1}$ is $\bigcup S_i$ for $1 \le i \le 5$ where $S_i = \{\pi + x_i + \pi' : \pi$ is a partition of θ_1 into parts $>$ the first part of x_i such that $\lambda + 1$ is not a part and $\alpha(\lambda + 1)$, $1 < \alpha < (2k - \lambda)$ can be repeated and π' is a partition of θ_2 into distinct parts $<$ the last part of x_i such that $\theta_1 + \theta_2 = n - x_i\}$ and $x_1 = (\frac{\lambda}{2} + 1) + \frac{\lambda}{2} + (\frac{\lambda}{2} - 1)$, $x_2 = (\frac{\lambda}{2} + 2) + (\frac{\lambda}{2} + 1) + \frac{\lambda}{2}$, $x_3 = (\frac{\lambda}{2} + 2) + \frac{\lambda}{2} + (\frac{\lambda}{2} - 1)$, $x_4 = (\frac{\lambda}{2} + 2) + (\frac{\lambda}{2} + 1) + (\frac{\lambda}{2} - 1)$, $x_5 = (\frac{\lambda}{2} + 2) + (\frac{\lambda}{2} + 1) + (\frac{\lambda}{2}) + (\frac{\lambda}{2} - 1)$. π and π' may not exist in some cases. We note that the partitions in the first two sets and the last set violate $S_{\frac{\lambda}{2}}$. We also observe that the last set will occur only when $k > a$ for if $k = a$ and $a - \frac{\lambda}{2} = 1$ then $2k - \lambda = 2a - \lambda = 2$ and $n < 2(\lambda + 1)$. For partitions π in the third and fourth sets we associate π^* in P'_B where π^* is the partition obtained by rearranging respectively $\pi + (\lambda + 1) + \frac{\lambda}{2} + \pi'$ and $\pi + (\lambda + 1) + (\frac{\lambda}{2} + 1) + \pi'$ in the decreasing order. Let $a - \frac{\lambda}{2} = 2$. Then $2(\lambda + 1) < n < (2k - \lambda - 1)(\lambda + 1)$. The set of partitions of n in P_A violating $S_{\frac{\lambda}{2}-1}$ is $\{\pi + (\frac{\lambda}{2} + 2) + (\frac{\lambda}{2} + 1) + \frac{\lambda}{2} + (\frac{\lambda}{2} - 1) + \pi' : \pi$ is a partition of θ_1 into parts $> (\frac{\lambda}{2}+2)$ such that $\alpha(\lambda+1)$, $1 \le \alpha < (2k-\lambda-1)$, $\alpha \ne 2$ can be repeated and π' is a partition of θ_2 into distinct parts $< \frac{\lambda}{2} - 1$ such that $\theta_1 + \theta_2 = n - 2(\lambda + 1)\}$. For a partition in the above set we associate π^* in P'_B where π^* is the partition obtained by rearranging $\pi + 2(\lambda + 1) + \pi'$ in the decreasing order.

Proceeding like this we arrive at the following step.

Step $\frac{\lambda}{2}$. Consider $S_1 : f_1 + \cdots + f_\lambda \le a - 1$. Since $f_i \le 1$ for all $1 \le i \le \lambda$, we have $f_1 + \cdots + f_\lambda \le \lambda$.

1) Suppose all the parts $1, 2, \ldots, \lambda$ appear in a partition of n and let $a - \frac{\lambda}{2} = \frac{\lambda}{2} - 1$. Then $(a - \frac{\lambda}{2})(\lambda + 1) < n < (2k - a - \frac{\lambda}{2} - 1)(\lambda + 1)$ becomes $(\frac{\lambda}{2} - 1)(\lambda + 1) < n < (a - \frac{\lambda}{2} + 1)(\lambda + 1) = (\frac{\lambda}{2})(\lambda + 1)$ because $a - \frac{\lambda}{2} = \frac{\lambda}{2} - 1$ implies $a = \lambda - 1$ which implies $k = a = \lambda - 1$. Since all the parts $1, 2, \ldots, \lambda$ appear in a partition of n the sum is $\ge \lambda + \cdots + 2 + 1 = (\frac{\lambda}{2})(\lambda + 1) > n$. Hence all the parts $1, 2, \ldots, \lambda$ cannot appear when $a - \frac{\lambda}{2} = \frac{\lambda}{2} - 1$.

   Let $a - \frac{\lambda}{2} < \frac{\lambda}{2} - 1$ and let all the parts $1, 2, \ldots, \lambda$ appear in a partition of n. Then such partitions will be of the form

$$\text{(6)} \quad \pi + \lambda + (\lambda - 1) + \cdots + 2 + 1 = \pi + \underbrace{(\lambda + 1) + \cdots + (\lambda + 1)}_{(\frac{\lambda}{2})\mathit{times}}$$

where π is a partition of $n - (\lambda + (\lambda - 1) + \cdots + 2 + 1)$ into parts $> \lambda$ such that multiples of $\lambda + 1$ can be repeated but $(a - \frac{\lambda}{2})(\lambda + 1)$ is not a part. Consider the partition

$$\pi + \left(a - \frac{\lambda}{2}\right)(\lambda + 1) + \underbrace{(\lambda + 1) + \cdots + (\lambda + 1)}_{(\lambda - a) times}$$

and rearrange the parts in decreasing order. Let the resulting partition be π^*. Since $(a - \frac{\lambda}{2})(\lambda + 1)$ is a part in π^*, clearly π^* does not belong to P_A but belongs to P_B and hence $\pi' \in P'_B$. We associate (6) to $\pi^* \in P'_B$.

2) Now, if $f_1 + \cdots + f_\lambda = \lambda - y, 1 \le y \le \lambda - a$ and if $x_1, \ldots, x_y$ are the parts which are left out with $1 \le x_1 < x_2 < \cdots < x_y \le \lambda$, then

$$\begin{aligned} &\lambda + (\lambda - 1) + \cdots + (x_y + 1) + (x_y - 1) \\ &\quad + \cdots + (x_1 + 1) + (x_1 - 1) + \cdots + 2 + 1 \\ &= \left(\frac{\lambda}{2} - y\right)(\lambda + 1) + (\lambda + 1 - x_1) + \cdots + (\lambda + 1 - x_y) \end{aligned} \tag{7}$$

If $a - \frac{\lambda}{2} < \frac{\lambda}{2} - y$, then the partitions of n violating S_1 are

$$\begin{aligned} &\pi + \lambda + (\lambda - 1) + \cdots + (x_y + 1) + (x_y - 1) \\ &\quad + \cdots + (x_1 + 1) + (x_1 - 1) + \cdots + 2 + 1 \\ &= \pi + \left(\frac{\lambda}{2} - y\right)(\lambda + 1) + (\lambda + 1 - x_1) + \cdots + (\lambda + 1 - x_y) \end{aligned} \tag{8}$$

where π is a partition of $n - (7)$ into parts $> \lambda$ such that multiples of $\lambda + 1$ can be repeated but $(a - \frac{\lambda}{2})(\lambda + 1)$ is not a part. Consider the partition

$$\begin{aligned} &\pi + \left(a - \frac{\lambda}{2}\right)(\lambda + 1) + \underbrace{(\lambda + 1) + \cdots + (\lambda + 1)}_{(\lambda - a - y) times} \\ &\quad + (\lambda + 1 - x_1) + \cdots + (\lambda + 1 - x_y) \end{aligned}$$

and rearrange the parts in decreasing order. Let the resulting partition be π^*. We associate (8) to $\pi^* \in P'_B$.

If $(a - \frac{\lambda}{2}) = \frac{\lambda}{2} - y$, then the partitions of n violating S_1 are

$$\begin{aligned} &\pi + \lambda + (\lambda - 1) + \cdots + (x_y + 1) + (x_y - 1) \\ &\quad + \cdots + (x_1 + 1) + (x_1 - 1) + \cdots + 2 + 1 \\ &= \pi + \left(\frac{\lambda}{2} - y\right)(\lambda + 1) + (\lambda + 1 - x_1) + \cdots + (\lambda + 1 - x_y) \end{aligned} \tag{9}$$

where π is a partition of

$$\begin{aligned} n_1 &= n - [\lambda + (\lambda - 1) + \cdots + (x_y + 1) + (x_y - 1) + \cdots + (x_1 + 1) \\ &\quad + (x_1 - 1) + \cdots + 2 + 1] \end{aligned}$$

into parts $> \lambda$ such that multiples of $\lambda + 1$ can be repeated but $(a - \frac{\lambda}{2})(\lambda + 1)$ is not a part. Since $(a - \frac{\lambda}{2})(\lambda + 1) < n < (2k - a - \frac{\lambda}{2} + 1)(\lambda + 1)$ we can write $n = (2k - a - \frac{\lambda}{2})(\lambda + 1) + \theta,\ \theta < \lambda + 1$. Also $n = n_1 + (\frac{\lambda}{2} - y)(\lambda + 1) + (\lambda + 1 - x_1) + \cdots + (\lambda + 1 - x_y)$. Hence

$$\begin{aligned} &n_1 - \theta + (\lambda + 1 - x_1) + \cdots + (\lambda + 1 - x_y) \\ &= (2k - a - \lambda + y)(\lambda + 1) \end{aligned}$$

Let $(\lambda + 1 - x_1) + \cdots + (\lambda + 1 - x_y) = \alpha$. Then $n_1 - \theta + \alpha = (2k - a - \lambda + y)(\lambda + 1) \leq (2k - 2a)(\lambda + 1)$ since $y \leq \lambda - a$. This gives $n_1 - \theta \leq 2(k - a)(\lambda + 1) - \alpha$. There are no partitions of n violating S_1 if $n_1 - \theta < 2(k - a)(\lambda + 1) - \alpha$. The partition (9) violates S_1 when $n_1 - \theta = 2(k - a)(\lambda + 1) - \alpha$. Consider the partition

$$\pi + \left(a - \frac{\lambda}{2}\right)(\lambda + 1) + (\lambda + 1 - x_1) + \cdots + (\lambda + 1 - x_y)$$

and rearrange the parts in decreasing order. Let the resulting partition be π^*. We associate (9) to $\pi^* \in P'_B$. The case $a - \frac{\lambda}{2} > \frac{\lambda}{2} - y$ is similar to the earlier ones.

Note. Among the deleted parts $x_1, \ldots, x_y$ if two of them say x_{i_1} and x_{i_2} are respectively equal to $\frac{\lambda}{2} + 1$ and $\frac{\lambda}{2}$ then in the associated partition, the parts

$$\begin{aligned} &(\lambda + 1 - x_{i_1}) + (\lambda + 1 - x_{i_2}) \\ &= \left[\lambda + 1 - \left(\frac{\lambda}{2} + 1\right)\right] + \left(\lambda + 1 - \frac{\lambda}{2}\right) = \frac{\lambda}{2} + \left(\frac{\lambda}{2} + 1\right) \end{aligned}$$

should be replaced by $\lambda + 1$.

Step $\frac{\lambda}{2} + 1$. Consider $S : f_1 + \cdots + f_{\lambda+1} \leq a - 1$. Here $f_i \leq 1$ for $1 \leq i \leq \lambda$ and $f_{\lambda+1} \leq 2k - a - \frac{\lambda}{2}$. Let $f_1 + \cdots + f_{\lambda+1} = x,\ x \geq \lambda$. Let $f_1 + \cdots + f_\lambda = \theta_1$ and $f_{\lambda+1} = \theta_2$ where $0 < \theta_2 \leq 2k - a - \frac{\lambda}{2}$. Then $x = \theta_1 + \theta_2$. We observe that $\theta_2 = 0$ case reduces to Step $\frac{\lambda}{2}$.

1) Let $f_{\lambda+1} = 1$. In this case either all the parts $1, 2, \ldots, \lambda$ appear or one of the parts say x can be deleted since $f_1 + \cdots + f_{\lambda+1} \geq \lambda$. Consider the partition

$$\pi + (\lambda + 1) + \lambda + \cdots + 2 + 1 \tag{10}$$

where π is a partition of $n - [(\lambda+1)+\lambda+\cdots+2+1]$ into parts $> \lambda+1$ such that multiples of $\lambda+1$ can be repeated but $(a-\frac{\lambda}{2})(\lambda+1)$ is not a part. If $(a-\frac{\lambda}{2}) = 1$ then partitions of the type (10) will not be there as $(a-\frac{\lambda}{2})(\lambda+1) = (\lambda+1)$ is not a part. Hence let $a-\frac{\lambda}{2} \geq 2$. Here $f_1+\cdots+f_{\lambda+1} = \lambda+1 > a-1$ and therefore partition (10) violates S. Clearly, $(10) = \pi + (\frac{\lambda}{2}+1)(\lambda+1)$.

Let $\frac{\lambda}{2}+1 = (a-\frac{\lambda}{2})m_1 + m_1'$ where $0 \leq m_1' < a-\frac{\lambda}{2}$. Now $m_1' < a-1$. Consider the partition

$$\underbrace{\pi+\left(a-\frac{\lambda}{2}\right)(\lambda+1)+\cdots+\left(a-\frac{\lambda}{2}\right)(\lambda+1)}_{m_1 times}+\underbrace{(\lambda+1)+\cdots+(\lambda+1)}_{(m_1')times}$$

$$+(\lambda+1-x_1)+\cdots+\ (\lambda+1-x_y)$$

and rearrange the parts in decreasing order. Let the resulting partition be π^*. We associate (10) to $\pi^* \in P_B'$.

If one of the parts say x among $1, 2, \ldots, \lambda$ is deleted then we consider the partition

$$(11) \quad \pi+(\lambda+1)+\lambda+\cdots+(x+1)+(x-1)+\cdots+2+1$$

where π is a partition of $n - [(\lambda+1)+\lambda+\cdots+(x+1)+(x-1)+\cdots+2+1]$ into parts $> \lambda+1$ such that multiples of $\lambda+1$ can be repeated but $(a-\frac{\lambda}{2})(\lambda+1)$ is not a part. Clearly partitions of the type (11) will not be there if $(a-\frac{\lambda}{2}) = 1$. Hence let $a-\frac{\lambda}{2} \geq 2$. Now

$$(11) = \pi + \left(\frac{\lambda}{2}\right)(\lambda+1)+(\lambda+1-x)$$

Let $\frac{\lambda}{2} = (a-\frac{\lambda}{2})m_2 + m_2'$ where $0 \leq m_2' < a-\frac{\lambda}{2}$. Consider the partition

$$\underbrace{\pi+\left(a-\frac{\lambda}{2}\right)(\lambda+1)+\cdots+\left(a-\frac{\lambda}{2}\right)(\lambda+1)}_{m_2 times}$$

$$+\underbrace{(\lambda+1)+\cdots+(\lambda+1)}_{m_2' times}+\lambda+1-x$$

and rearrange the parts in decreasing order. Let the resulting partition be π^*. We associate (11) to $\pi^* \in P_B'$.

2) More generally, let $f_{\lambda+1} = \theta_2$ where $0 < \theta_2 \leq 2k-a-\frac{\lambda}{2}$. In this case we have the following (θ_2+1) steps.

i) All the parts $1, 2, \ldots, \lambda$ appear. Consider the partition

$$(12) \qquad \pi + \underbrace{(\lambda+1) + \cdots + (\lambda+1)}_{\theta_2 times} + \lambda + \cdots + 2 + 1$$

where π is a partition of $n - [\theta_2(\lambda+1) + \lambda + \cdots + 2 + 1]$ into parts $> \lambda+1$ such that multiples of $\lambda + 1$ can be repeated but $(a - \frac{\lambda}{2})(\lambda + 1)$ is not a part. Clearly partitions of the type (12) will not be there if $(a - \frac{\lambda}{2}) = 1$. Hence let $a - \frac{\lambda}{2} \geq 2$. Now

$$(12) = \pi + \left(\frac{\lambda}{2} + \theta_2\right)(\lambda + 1)$$

Let $\frac{\lambda}{2} + \theta_2 = (a - \frac{\lambda}{2})m_3 + m_3'$ where $O \leq m_3' < a - \frac{\lambda}{2}$. Consider the partition

$$\pi + \underbrace{\left(a - \frac{\lambda}{2}\right) + \cdots + \left(a - \frac{\lambda}{2}\right)}_{m_3 times} + \underbrace{(\lambda+1) + \cdots + (\lambda+1)}_{m_3' times}$$

and rearrange the parts in decreasing order. Let the resulting partition be π^*. We associate (12) to $\pi^* \in P_B'$.

Proceeding like this we arrive at the $(\theta_2 + 1)$th step.

$\theta_2 + 1)$ Here θ_2 parts among $1, 2, \ldots, \lambda$ say $x_1, \ldots, x_{\theta_2}$ with $1 \leq x_1 < x_2 < \cdots < x_{\theta_2} \leq \lambda$ are deleted. Consider the partition

$$\begin{aligned}
&\pi + \underbrace{(\lambda+1) + \cdots + (\lambda+1)}_{\theta_2 times} + \lambda + \cdots + (x_{\theta_2} + 1) \\
&\qquad + (x_{\theta_2} - 1) + \cdots + (x_1 + 1) + (x_1 - 1) + \cdots + 2 + 1 \\
&\qquad = \pi + \left(\frac{\lambda}{2}\right)(\lambda+1) + (\lambda + 1 - x_1) + \cdots + (\lambda + 1 - x_{\theta_2}) \\
(13) &\qquad = \pi + Q, \quad (say)
\end{aligned}$$

where π is a partition of $n - Q$ into parts $> \lambda + 1$ such that multiples of $\lambda + 1$ can be repeated but $(a - \frac{\lambda}{2})(\lambda + 1)$ is not a part. Clearly partitions of the type (13) will not be there if $(a - \frac{\lambda}{2}) = 1$. Hence let $a - \frac{\lambda}{2} \geq 2$.

Let $\frac{\lambda}{2} = (a - \frac{\lambda}{2})m_4 + m_4'$ where $0 \leq m_4' < a - \frac{\lambda}{2}$. Consider the partition

$$\pi + \underbrace{\left(a - \frac{\lambda}{2}\right) + \cdots + \left(a - \frac{\lambda}{2}\right)}_{m_4 times} + \underbrace{(\lambda+1) + \cdots + (\lambda+1)}_{m_4' times}$$

$$+ (\lambda + 1 - x_1) + \cdots + (\lambda + 1 - x_{\theta_2})$$

and rearrange the parts in decreasing order. Let the resulting partition be π^*. We associate (13) to $\pi^* \in P'_B$.

Now let $f_1 + \cdots + f_{\lambda+1} < \lambda$, i.e., $f_1 + \cdots + f_{\lambda+1} = \lambda - y,\ 1 \leq y \leq \lambda - a$ and let $f_{\lambda+1} = \alpha,\ 1 \leq \alpha \leq 2k - a - \frac{\lambda}{2}$. Let $x_1, \ldots, x_{y+\alpha}$ be the parts deleted among $1, 2, \ldots, \lambda$ with $1 \leq x_1 < x_2 < \cdots < x_{y+\alpha} \leq \lambda$. Then

$$\begin{aligned}
&\underbrace{(\lambda+1) + \cdots + (\lambda+1)}_{\alpha times} + \lambda + \cdots + (x_{y+\alpha} + 1) + (x_{y+\alpha} - 1) \\
&\quad + \cdots + (x_1 + 1) + (x_1 - 1) + \cdots + 2 + 1 \\
&\quad = \alpha(\lambda+1) + \left(\frac{\lambda}{2} - \alpha - y\right)(\lambda+1) + (\lambda + 1 - x_1) \\
&\quad + \cdots + (\lambda + 1 - x_{y+\alpha}) \\
&\quad = \left(\frac{\lambda}{2} - y\right)(\lambda+1) + (\lambda + 1 - x_1) + \cdots + (\lambda + 1 - x_{y+\alpha})
\end{aligned}$$

If $a - \frac{\lambda}{2} < \frac{\lambda}{2} - y$ then the partitions of n violating S are

$$\begin{aligned}
&\pi + \underbrace{(\lambda+1) + \cdots + (\lambda+1)}_{\alpha times} + \lambda + \cdots + (x_{y+\alpha} + 1) \\
&\quad + (x_{y+\alpha} - 1) + \cdots + (x_1 + 1) + (x_1 - 1) + \cdots + 2 + 1 \\
&\quad = \pi + \left(\frac{\lambda}{2} - y\right)(\lambda+1) + (\lambda + 1 - x_1) + \cdots + (\lambda + 1 - x_{y+\alpha}) \\
(14) \qquad &\quad = \pi + Q_1 \ (say)
\end{aligned}$$

where π is a partition of $n - Q_1$ into parts $> \lambda + 1$ such that multiples of $\lambda + 1$ can be repeated but $(a - \frac{\lambda}{2})(\lambda + 1)$is not a part. Clearly partitions of the type (14) will not be there if $a - \frac{\lambda}{2} = 1$. Hence let $a - \frac{\lambda}{2} \geq 2$. Consider the partition

$$\begin{aligned}
&\pi + \left(a - \frac{\lambda}{2}\right)(\lambda+1) + \underbrace{(\lambda+1) + \cdots + (\lambda+1)}_{(\lambda-a-y) times} \\
&\quad + (\lambda + 1 - x_1) + \cdots + (\lambda + 1 - x_{y+\alpha})
\end{aligned}$$

and rearrange the parts in decreasing order. Let the resulting partition be π^*. We associate (14) to $\pi^* \in P'_B$. If $a - \frac{\lambda}{2} = \frac{\lambda}{2} - y$, then the partitions of n violating S are

$$\begin{aligned}
&\pi + \underbrace{(\lambda+1) + \cdots + (\lambda+1)}_{\alpha times} + \lambda + \cdots + (x_{y+\alpha} + 1) \\
&\quad + (x_{y+\alpha} - 1) + \cdots + (x_1 + 1) + (x_1 - 1) + \cdots + 2 + 1
\end{aligned}$$

$$= \pi + \left(\frac{\lambda}{2} - y\right)(\lambda + 1) + (\lambda + 1 - x_1) + \cdots + (\lambda + 1 - x_{y+\alpha})$$

$$= \pi + Q_2 \text{ (say)} \tag{15}$$

where π is a partition of $n_1 = n - Q_2$ into parts $> \lambda + 1$ such that multiples of $\lambda + 1$ can be repeated but $(a - \frac{\lambda}{2})(\lambda + 1)$ is not a part. Clearly partitions of the type (15) will not be there if $a - \frac{\lambda}{2} = 1$. Hence let $a - \frac{\lambda}{2} \geq 2$. Since $(a - \frac{\lambda}{2})(\lambda + 1) < n < (2k - a - \frac{\lambda}{2} + 1)(\lambda + 1)$ we can write $n = (2k - a - \frac{\lambda}{2})(\lambda + 1) + \theta$, $\theta < \lambda + 1$. Also $n = n_1 + (\frac{\lambda}{2} - y)(\lambda + 1) + (\lambda + 1 - x_1) + \cdots + (\lambda + 1 - x_{y+\alpha})$. Hence

$$\begin{aligned} &n_1 - \theta + (\lambda + 1 - x_1) + \cdots + (\lambda + 1 - x_{y+\alpha}) \\ &\quad = (2k - a - \lambda + y)(\lambda + 1) \end{aligned}$$

Let $(\lambda + 1 - x_1) + \cdots + (\lambda + 1 - x_{y+\alpha}) = \alpha'$. Then $n_1 - \theta + \alpha' = (2k - a - \lambda + y)(\lambda + 1) \leq (2k - 2a)(\lambda + 1)$ since $y \leq \lambda - a$. This gives $n_1 - \theta \leq 2(k - a)(\lambda + 1) - \alpha'$. There are no partitions of n violating S if $n_1 - \theta < 2(k - a)(\lambda + 1) - \alpha'$. The partition (15) violates S when $n_1 - \theta = 2(k - a)(\lambda + 1) - \alpha'$. Consider the partition

$$\pi + \left(a - \frac{\lambda}{2}\right)(\lambda + 1) + (\lambda + 1 - x_1) + \cdots + (\lambda + 1 - x_{y+\alpha})$$

and rearrange the parts in decreasing order. Let the resulting partition be π^*. We associate (15) to $\pi^* \in P_B'$. As before there will no partitions of n violating S when $a - \frac{\lambda}{2} > \frac{\lambda}{2} - y$.

Step $\frac{\lambda}{2} + 2$. Let $(a - \frac{\lambda}{2})(\lambda + 1) < n < (2k - a - \frac{\lambda}{2} + 1)(\lambda + 1)$. We first prove that $n < k(\lambda + 1)$. Let $k = a + \theta$, $0 \leq \theta \leq \frac{\lambda}{2} - 2$. Then $k - a - \frac{\lambda}{2} + 1 = \theta - \frac{\lambda}{2} + 1 \leq \frac{\lambda}{2} - 2 - \frac{\lambda}{2} + 1 = -1 < 0$. Hence $2k - a - \frac{\lambda}{2} + 1 < k$ implies $(2k - a - \frac{\lambda}{2} + 1)(\lambda + 1) < k(\lambda + 1)$ which implies $n < k(\lambda + 1)$.

Next we prove that if a partition violates S^* then it violates one of the conditions on $f's$. If a partition violates S^* then there exists a partition

$$n = b_1 + \cdots + b_i + \cdots + b_{i+k-1} + \cdots + b_k + \cdots + b_s \tag{16}$$

and an integer i with $b_i - b_{i+k-1} < \lambda + 1$. If $b_{i+k-1} \geq \lambda + 1$ then the number being partitioned is

$$\geq (\lambda + 1) + \cdots + (\lambda + 1) + \cdots \geq k(\lambda + 1) > n$$

Thus let $b_{i+k-1} < \lambda + 1$. If $b_i < \lambda + 1$ then (16) contains at least k parts $\leq \lambda$ and hence $f_1 + \cdots + f_\lambda \geq k > a - 1$ which implies that such a partition violates S_1.

Let $b_{i+k-1} < \lambda + 1$ and $b_i \geq \lambda + 1$. Since $n = (2k - a - \frac{\lambda}{2})(\lambda + 1) + \theta$, $\theta < \lambda + 1$, the number of parts $\geq \lambda + 1$ among $b_i, \ldots, b_{i+k-1}$ is $\geq 2k - a - \frac{\lambda}{2}$. If

$2k - a - \frac{\lambda}{2}$ parts are equal to $\lambda + 1$ then $f_{\lambda+1} = 2k - a - \frac{\lambda}{2}$ and the remaining $k - (2k - a - \frac{\lambda}{2}) = a + \frac{\lambda}{2} - k$ parts are $\le \lambda$ and hence $f_1 + \cdots + f_\lambda + f_{\lambda+1} = a + \frac{\lambda}{2} - k + 2k - a - \frac{\lambda}{2} = k > a - 1$ and such a partition violates S.

If the number of parts among $1, 2, \ldots, \lambda+1$ is $\ge a$ then the partition violates S or S_1. Let β denote the number of parts among $1, 2, \ldots, \lambda+1$. Then $1 \le \beta \le a-1$. Let α denote the number of parts $> \lambda + 1$ so that $k - a + 1 \le \alpha \le k - 1$. Then the number being partitioned is

$$(\lambda + x_\alpha) + \cdots + (\lambda + x_1) + y_1 + \cdots + y_\beta \tag{17}$$

Since $\lambda + x_\alpha - y_\beta < \lambda + 1$, we have $x_\alpha = y_\beta$. Now $x_1 \ge 2, x_2 \ge 3, \ldots, x_\alpha \ge \alpha + 1$. Thus $y_\beta \ge \alpha + 1, \ldots, y_1 \ge \alpha + \beta = k$. Hence (17) is

$$\ge (\lambda + \alpha + 1) + \cdots + (\lambda + 2) + (\alpha + \beta) + \cdots + (\alpha + 1)$$

$$= \alpha(\lambda + 1) + \frac{[(\alpha + \beta)(\alpha + \beta + 1)]}{2} = \alpha(\lambda + 1) + \frac{k(k+1)}{2}$$

i) Let $\beta = 1$. In this case (17) is $\ge (k-1)(\lambda+1) + \frac{k(k+1)}{2}$. Now we prove that

$$(k-1)(\lambda+1) + \frac{k(k+1)}{2} > \left(2k - a - \frac{\lambda}{2} + 1\right)(\lambda+1) \qquad \textit{for } \lambda > 4.$$

In fact, if $k = \frac{\lambda}{2} + \theta,\ 0 < \theta < \frac{\lambda}{2}$ then r.h.s–l.h.s is equal to

$$\frac{[(\lambda - 2\theta)^2 + 2(\lambda - 2\theta)]}{8} + \left(\frac{\lambda}{2} - 2\right)(\lambda + 1)$$

Proceeding like this we arrive at the $(a-1)$th step.

$a-1$) Let $\beta = a - 1$. Now (17) is $\ge (k - a + 1)(\lambda + 1) + \frac{k(k+1)}{2}$. We prove that

$$(k - a + 1)(\lambda + 1) + \frac{k(k+1)}{2} > \left(2k - a - \frac{\lambda}{2} + 1\right)(\lambda + 1)$$

This follows from r.h.s–l.h.s $= \frac{[(\lambda - 2\theta)^2 + 2(\lambda - 2\theta)]}{8}$, where $k = \frac{\lambda}{2} + \theta,\ 0 < \theta < \frac{\lambda}{2}$. Thus for $(a - \frac{\lambda}{2})(\lambda + 1) < n < (2k - a - \frac{\lambda}{2} + 1)(\lambda + 1)$, if a partition violates S^* then we have proved that it violates either S or S_1.

In the procedure explained above we add the following condition: For partitions violating $S_{\frac{\lambda}{2}-2x}$, $1 \le x \le [\frac{(\frac{\lambda}{2}-2)}{2}]$ =integral part of $\frac{(\frac{\lambda}{2}-2)}{2}$, we replace $(\frac{\lambda}{2} + (2x + 1)) + (\frac{\lambda}{2} + 2x) + (\frac{\lambda}{2} - (2x - 1)) + (\frac{\lambda}{2} - 2x)$ by $(\lambda + 1) + (\lambda + 1)$.

Proofs of (4) and (5). (4) [resp.(5)] is the case $k = a + 1$ [resp. $k = a + 2$] of the conjecture. As in the proof of (3) we can show that every partition in P'_B has an associate in P'_A except $(2k - a - \frac{\lambda}{2} + 1)(\lambda + 1)$ for (4) and $(2k - a - \frac{\lambda}{2} + 1)(\lambda + 1) + 1$ for (5).

We now illustrate our procedure by considering a numerical example.

Example. Let $\lambda = 6,\ k = 5,\ a = 4$,

$$n = \frac{[(k + \lambda - a + 1)(k + \lambda - a)]}{2} + (k - \lambda + 1)(\lambda + 1) = 28.$$

Because of the lengthy procedure, we illustrate only for $n = 28$. Since $P_{A_{6,5,4}}(28) = 164$ and $P_{B_{6,5,4}}(28) = 165$, we write only P'_A and P'_B.

TABLE

$P'_{A_{6,5,4}}(28)$	$P'_{B_{6,5,4}}(28)$
21 + 4 + 3, 20 + 4 + 3 + 1, 19 + 4 + 3 + 2,	28, 21 + 7, 20 + 7 + 1,
18 + 5 + 3 + 2, 18 + 4 + 3 + 2 + 1,	19 + 7 + 2, 18 + 7 + 3, 18 + 7 + 2 + 1,
17 + 5 + 4 + 2, 17 + 5 + 3 + 2 + 1,	17 + 7 + 4, 17 + 7 + 3 + 1, 16 + 7 + 3 + 2,
16 + 6 + 3 + 2 + 1, 16 + 5 + 4 + 3,	16 + 7 + 5, 16 + 7 + 4 + 1, 15 + 7 + 6,
16 + 5 + 4 + 2 + 1, 15 + 6 + 4 + 3,	15 + 7 + 4 + 2, 15 + 7 + 5 + 1, 14 + 7 + 7,
15 + 6 + 4 + 2 + 1, 15 + 5 + 4 + 3 + 1,	14 + 7 + 6 + 1, 14 + 7 + 5 + 2, 13 + 8 + 7,
14 + 6 + 5 + 2 + 1, 14 + 5 + 4 + 3 + 2,	13 + 7 + 5 + 3, 13 + 7 + 6 + 2,
14 + 6 + 4 + 3 + 1, 13 + 8 + 4 + 3,	13 + 7 + 7 + 7 + 1, 12 + 9 + 7,
13 + 6 + 5 + 3 + 1, 13 + 6 + 4 + 3 + 2,	12 + 8 + 7 + 1, 12 + 7 + 5 + 4,
13 + 5 + 4 + 3 + 2 + 1, 12 + 9 + 4 + 3,	12 + 7 + 6 + 3, 12 + 7 + 7 + 2,
12 + 8 + 4 + 3 + 1, 12 + 6 + 5 + 4 + 1,	11 + 9 + 7 + 1, 11 + 8 + 7 + 2,
12 + 6 + 5 + 3 + 2, 12 + 6 + 4 + 3 + 2 + 1,	11 + 7 + 7 + 3, 11 + 7 + 6 + 4,
11 + 9 + 4 + 3 + 1, 11 + 8 + 4 + 3 + 2,	10 + 9 + 7 + 2, 10 + 8 + 7 + 3,
11 + 6 + 5 + 3 + 2 + 1, 11 + 6 + 5 + 4 + 2,	10 + 8 + 7 + 2 + 1, 10 + 7 + 6 + 5,
10 + 9 + 4 + 3 + 2, 10 + 8 + 5 + 3 + 2,	10 + 7 + 7 + 4, 9 + 8 + 7 + 4,
10 + 8 + 4 + 3 + 2 + 1, 10 + 6 + 5 + 4 + 3,	9 + 8 + 7 + 3 + 1, 9 + 7 + 7 + 5,
10 + 6 + 5 + 4 + 2 + 1, 9 + 8 + 5 + 4 + 2,	8 + 7 + 7 + 6
9 + 8 + 5 + 3 + 2 + 1, 9 + 6 + 5 + 4 + 3 + 1,	
8 + 6 + 5 + 4 + 3 + 2	

a) The partitions enumerated by $A_{6,5,4}(28)$ violating S_3 are
{21+4+3, 20+4+3+1, 19+4+3+2, 18+4+3+2+1, 16+5+4+3, 15+6+4+3, 15+5+4+3+1, 14+5+4+3+2, 14+6+4+3+1, 13+8+4+3, 13+6+4+3+2, 12+9+4+3, 12+8+4+3+1, 11+9+4+3+1, 11+8+4+3+2, 10+9+4+3+2, 10+8+4+3+2+1}
for which their associates in P'_B are respectively
{21+7, 20+7+1, 19+7+2, 18+7+2+1, 16+7+5, 15+7+6, 15+7+5+1, 14+7+5+2, 14+7+6+1, 13+8+7, 13+7+6+2, 12+9+7, 12+8+7+1, 11+9+7+1, 11+8+7+2, 10+9+7+2, 10+8+7+2+1}

b) The partitions enumerated by $A_{6,5,4}(28)$ violating S_2 are $\{18+5+3+2, 17+5+4+2, 17+5+3+2+1, 16+5+4+2+1, 10+8+5+3+2, 9+8+5+4+2, 9+8+5+3+2+1\}$

for which their associates in P_B' are respectively $\{18+7+3, 17+7+4, 17+7+3+1, 16+7+4+1, 10+8+7+3, 9+8+7+4, 9+8+7+3+1\}$

c) The partitions enumerated by $A_{6,5,4}(28)$ violating S_1 are $\{16+6+3+2+1, 15+6+4+2+1, 14+6+5+2+1, 13+6+5+3+1, 13+5+4+3+2+1, 12+6+5+4+1, 12+6+5+3+2, 12+6+4+3+2+1, 11+6+5+3+2+1, 11+6+5+4+2, 10+6+5+4+3, 10+6+5+4+2+1, 9+6+5+4+3+1, 8+6+5+4+3+2\}$

for which their associates in P_B' are respectively $\{16+7+3+2, 15+7+4+2, 14+7+7, 13+7+5+3, 13+7+7+1, 12+7+5+4, 12+7+6+3, 12+7+7+2, 11+7+7+3, 11+7+6+4, 10+7+6+5, 10+7+7+4, 9+7+7+5, 8+7+7+6\}$

d) Since $(a-\frac{\lambda}{2})(\lambda+1)=7$ is not a part in the partitions enumerated by $A_{6,5,4}(28)$, there will be no partitions violating S.

e) The partition $28=[2(5)-4-3+1](7)\in P'_{B_{6,5,4}}(28)$ has no associate in $P'_{A_{6,5,4}}(28)$ while all other partitions have. Hence

$$P'_{B_{6,5,4}}(28) = P'_{A_{6,5,4}}(28)+1$$

Case 2. Let λ be odd. Then

18. $B_{\lambda,k,a}(n) = A_{\lambda,k,a}(n)$ for $n \le \lambda$

19. $B_{\lambda,k,a}(\lambda+1) = A_{\lambda,k,a}(\lambda+1)$

20. $B_{\lambda,k,a}(n) = A_{\lambda,k,a}(n)$ for $n \le (2a-\lambda)[\frac{(\lambda+1)}{2}]$

21. $B_{\lambda,k,a}(n) = A_{\lambda,k,a}(n)$ for $(2a-\lambda)[\frac{(\lambda+1)}{2}] < n < (4k-2a-\lambda+2)[\frac{(\lambda+1)}{2}]$

22. $B_{\lambda,k,a}[(4k-2a-\lambda+2)\frac{(\lambda+1)}{2}]$
$= A_{\lambda,k,a}[(4k-2a-\lambda+2)\frac{(\lambda+1)}{2}]+1$ when $k=a+1$

23. $B_{\lambda,k,a}\{[(4k-2a-\lambda+2)][\frac{(\lambda+1)}{2}]+1\}$
$= A_{\lambda,k,a}\{[(4k-2a-\lambda+2)][\frac{(\lambda+1)}{2}]+1\}+1$ when $k=a+2$

Identities (18)–(20) have been proved in [2]. Let S^{**} denote the condition 'no parts $\equiv 0(mod\ \lambda+1)$ are repeated'. A partition is in $P_A'(n)$ implies that it violates one of the conditions $S_1, \ldots, S_{\frac{\lambda+1}{2}}, S, S^*, S^{**}$. A proof similar to that of Step $\frac{\lambda}{2}+2$ of even λ will show that partitions violating S^* will also violate S_1. From the definition of $A_{\lambda,k,a}(n)$ for odd λ we note that S reduces to S_1 since $f_{\lambda+1}=0$. To prove (21)–(23) we enumerate the partitions in P_A violating $S_{\frac{(\lambda+1)}{2}}, \ldots, S_1, S^{**}$ on a line similar to the even case and also give the bijection of $P_A'(n)$ onto $P_B'(n)$.

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Integrity of $P_2 \times P_n$

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The vertex Integrity, $I(G)$, of a graph G is defined as

$$I(G) = min\{| S | + m(G - S) \mid S \subsetneq V(G)\}$$

where $m(G - S)$ is the order of the largest component of $G - S$. In this paper, we compute $I(P_2 \times P_n)$, the vertex integrity of the Cartesian product of P_2 and P_n.

Introduction

Computer or communication networks are so designed that they do not easily get disrupted under external attack. Moreover, they should be easily reconstructible, if disrupted. These desirable properties of networks can be measured by parameters like connectivity, toughness, integrity and tenacity. The concept of integrity was introduced by Barefoot, Entringer and Swart in [3] and it was further studied by Bagga, Beineke, Goddard, Lipman and Pippert (see [1], [2], [4], [7], [8]). Let G be a graph, $S \subsetneq V(G)$. We define

$$I(S) = | S | + m(G - S),$$

where $m(G - S)$ is the order of the largest component of $G - S$.

The vertex integrity, $I(G)$, is defined as

$$I(G) = min\{I(S)/S \subsetneq V(G)\}.$$

A set for which the above minimum is attained is called an **I-set** and an I-set with smallest cardinality is called a **minimum I-set.**

Integrity of paths was found by Barefoot, Entringer and Swart in [2].

Theorem 1. $I(P_n) = \lceil 2\sqrt{n+1} \, \rceil - 2$, *where* $\lceil x \rceil$ *denotes the smallest integer greater than or equal to* x.

In [2], a formula for $I(P_2 \times P_n)$ is given. According to this formula, $I(P_2 \times P_{10}) = 9$. However, it can be easily proved that $I(P_2 \times P_{10}) = 10$. In this paper, we give a correct formula for $I(P_2 \times P_{10})$.

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Notations.

The vertices of $P_2 \times P_n$ are labelled as follows:

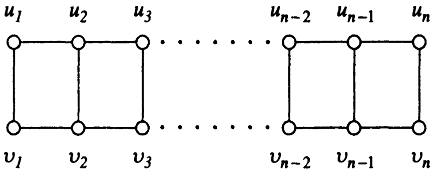


Figure 1

Henceforth, we denote the graph $P_2 \times P_n$ by G. In all diagrams, solid circles denote the vertices of the set S and hollow circles will denote vertices of $G - S$. The main technique of the proof is as follows. We usually want to prove that a minimal *I-set* has some properties. We assume the contrary and produce a subset S' of $V(G)$ with either $I(S') < I(S)$ or $I(S') = I(S)$ but $|S'| < |S|$. This contradicts with the fact that S is a minimum I-set. We now prove

Lemma 2. *If S is a minimum I-set with smallest number of components of order $m = m(G - S)$ in $G - S$, then $\{u_i, u_{i+1}\} \not\subseteq S$ and $\{v_i, v_{i+1}\} \not\subseteq S$ for $i = 1, 2, \ldots, n - 1$.*

Proof. Assume the contrary and suppose $\{v_i, v_{i+1}\} \subseteq S$ for some i. If $u_i \in S$, then $S' = S - \{v_i\}$ is an I-set, a contradiction to minimality of S. Hence $u_i \notin S$. Similarly, $u_{i+1} \notin S$. If $i = 1$, then $S' = S - \{v_1\}$ is an I-set, a contradiction. So $i \neq 1$. Similarly $i \neq n - 1$. Thus $2 \leq i \leq n - 2$. If $v_{i-1} \in S$, then $S' = S - \{v_i\}$ is an I-set, a contradiction to minimality of S. Hence $v_{i-1} \notin S$. Similarly $v_{i+2} \notin S$. If $u_{i-1} \notin S$, then $S' = S - \{v_i\}$ is an I-set, a contradiction. So $u_{i-1} \in S$. Similarly, $u_{i+2} \in S$. We have following situation:

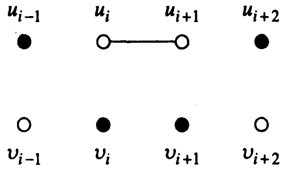
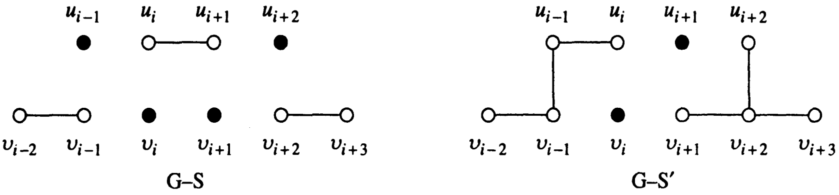


Figure 2

Suppose $\{v_{i-1}\}$ is a component of $G - S$ of order one. If $i = 2$, then $S' = S - \{u_1, v_2\} \cup \{u_2\}$ is an I-set, a contradiction. Hence $i \geq 3$. If $m \geq 3$, then $S' = S - \{v_i\}$ is an I-set, a contradiction. Therefore, $m = 2$. If v_{i+2} belongs to a component of $G - S$ of order two, then $S' = S - \{v_i, u_{i+2}\} \cup \{u_{i+1}\}$ is an I-set, a contradiction. Hence $\{v_{i+2}\}$ is a component of $G - S$ of order one. If $u_{i-2} \in S$, then $S' = S - \{u_{i-2}\}$ is an I-set, a contradiction. So $u_{i-2} \notin S$. If

$\{u_{i-2}\}$ is a component of $G - S$ of order one, then $S' = S - \{u_{i-1}, v_{i+1}\} \cup \{u_i\}$ is an I-set, a contradiction. Hence u_{i-2} belongs to a component of $G - S$ of order two. But now $S' = S - \{v_{i-2}, v_{i+1}\}$ is an I-set, a contradiction. Thus v_{i-1} belongs to a component of $G - S$ of order two or more. If $\{v_{i+2}\}$ is a component of $G - S$ of order one, then $S' = S - \{u_{i-1}, v_{i+1}\} \cup \{u_i\}$ is an I-set, a contradiction. Hence v_{i+2} belongs to a component of $G - S$ of order two or more. But then $S' = S - \{u_{i-1}, v_{i+1}, u_{i+2}\} \cup \{u_{i+1}\}$ is an I-set, a contradiction. This is illustrated in the diagram below:



G–S G–S′

Figure 3

This completes the proof. □

Remark 3. Let S be a minimum I-set for $G = P_2 \times P_n, n \geq 4$. If $m = 1$, then by minimality of S, exactly one of $\{u_1, v_1\}$ and exactly one of $\{u_n, v_n\}$ is in S. Suppose $u_1 \in S, v_n \in S$. Then $S' = S - \{u_1, v_n\}$ is an I-set, a contradiction. Hence $m \geq 2$.

Lemma 4. *If S is a minimum I-set with smallest number of components of order $m = m(G - S)$, then there does not exist any component of order one in $G - S$.*

Proof. Suppose $\{u_i\}$ is a component of order one in $G - S$. If $i = 1$, then $v_1, u_2 \in S$. If $v_2 \in S$, then $S' = S - \{v_2\}$ is an I-set, a contradiction. Hence $v_2 \notin S$. Suppose $\{v_2\}$ is a component of order one in $G - S$. By Remark 3, $m \geq 2$. But now $S' = S - \{v_1\}$ is an I-set, a contradiction. If v_2 belongs to a component of $G - S$ of order two, then both u_2 and u_3 are in S, a contradiction to Lemma 2. Hence order of component of $G - S$ that contains v_2 is three or more and $u_3 \notin S$. If order of component that contains v_2 is less than m, then $S' = S - \{v_1\}$ is an I-set, a contradiction. Hence v_2 belongs to a component of order m in $G - S$, $m \geq 3$. But now $S' = S - \{v_1\} \cup \{v_2\}$ is minimum I-set such that the number of components of order m in $G - S'$ is less than the number of components of order m in $G - S$, a contradiction. Hence $i \neq 1$. Similarly, $i \neq n$ and hence $2 \leq i \leq n - 1$. Since $\{u_i\}$ is a component of order one, $u_{i-1}, u_{i+1}, v_i \in S$. By Lemma 2, none of the vertices $u_{i-2}, v_{i-1}, v_{i+1}$, and u_{i+2} are in S. Suppose u_{i-2} and v_{i-1} are in the same component of $G - S$. If u_{i+2} and v_{i+1} are also in one component of $G - S$, then $S' = S - \{u_{i-1}, u_{i+1}\} \cup \{u_i\}$ is an I-set, a contradiction. Thus u_{i+2} and v_{i+1} are in different components of $G - S$. In this case, $v_{i+2} \in S$. If $\{u_{i+2}\}$ is a component of order one in $G - S$, then $S' = S - \{u_{i+1}\}$ is an I-set, a contradiction. Hence

u_{i+2} belongs to a component of order two or more in $G - S$. Since $v_{i+2} \in S$, by Lemma 2, $v_{i+3} \notin S$. Thus we have the following situation:

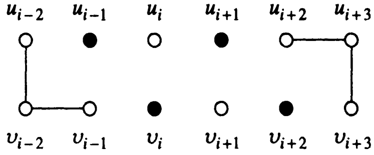


Figure 4

But now $S' = S - \{u_{i-1}, v_{i+2}\}$ is an I-set, a contradiction. Hence u_{i-2} and v_{i-1} belong to two different components of $G - S$ and also u_{i+2} and v_{i+1} are in two different components of $G-S$. Thus v_{i-2} and v_{i+2} are in S. If $m \geq 3$, then $S-\{v_i\}$ is an I-set, a contradiction. By Remark 3, $m \neq 1$. Hence $m = 2$. Suppose u_{i-2} belongs to a component of $G - S$ of order two, then $S' = S - \{v_{i-2}, v_i\} \cup \{v_{i-1}\}$ is an I-set, a contradiction. Therefore, $\{u_{i-2}\}$ is a component of order one in $G - S$. Similarly, $\{u_{i+2}\}$ is also a component of order one in $G - S$. But now $S' = S - \{u_{i-1}, u_{i+1}\} \cup \{u_i\}$ is an I-set, a contradiction. This completes the proof. ☐

Remark 5. Suppose S is a minimum I-set with smallest number of components of order m and let $u_i \in S$. By Lemma 2, u_{i-1} and u_{i+1} are not in S. Since S is minimum I-set, at least one but at most two of $\{v_{i-1}, v_i, v_{i+1}\}$ are in S. By Lemma 2, both v_{i-1} and v_i are not in S. Similarly, v_i, v_{i+1} are not simultaneously in S. By Lemma 4, v_{i-1} and v_{i+1} are not simultaneously in S. Hence exactly one of $\{v_{i-1}, v_i, v_{i+1}\}$ is in S. It follows that $|S| = 2t$ for $t \in \mathbb{N}$ and $G - S$ has $t + 1$ components. Hence

$$I(S) \geq 2t + \left\lceil \frac{2n-2t}{t+1} \right\rceil = 2t - 2 + \left\lceil \frac{2n+2}{t+1} \right\rceil.$$

We now prove:

Lemma 6. *For $t, n \in \mathbb{N}$, let $f(t) = 2t - 2 + \lceil \frac{2n+2}{t+1} \rceil$. Then the minimum value of $f(t)$ is given by*

$$\min f(t) = \begin{cases} 4r-3, & if \quad r^2 \leq n < r^2 + \lfloor \frac{r}{2} \rfloor \\ 4r-2, & if \quad r^2 + \lfloor \frac{r}{2} \rfloor \leq n < r^2 + r \\ 4r-1, & if \quad r^2 + r \leq n < r^2 + \lfloor \frac{3r+1}{2} \rfloor \\ 4r, & if \quad r^2 + \lfloor \frac{3r+1}{2} \rfloor \leq n < (r+1)^2. \end{cases}$$

Proof. Let $g(x) = 2x - 2 + \frac{2n+2}{x+1}$, where x is positive real number. Then $g(x)$ achieves its minimum at $x = \sqrt{n+1} - 1$. Moreover, $g(x)$ is increasing when

$x > \sqrt{n+1}-1$ and decreasing when $0 < x < \sqrt{n+1}-1$. Let $\ell = \lceil\sqrt{n+1}\rceil-1$ and $\ell' = \lfloor\sqrt{n+1}\rfloor - 1$. Then

$$g(s) \geq g(\ell) \quad \text{for all } s \in \mathbb{N}, s \geq \ell$$

and

$$g(s) \geq g(\ell') \quad \text{for all } s \in \mathbb{N}, s \leq \ell'.$$

It follows that

$$f(s) \geq f(\ell) \quad \text{for all } s \in \mathbb{N}, s \geq \ell$$
$$f(s) \geq f(\ell') \quad \text{for all } s \in \mathbb{N}, s \leq \ell'.$$

Hence the minimum of $f(t)$ is minimum of $\{f(\ell), f(\ell')\}$.

Some straightforward calculations then give the value of this minimum, as stated above. □

We can now prove the main result.

Theorem 7.

$$I(G) = \begin{cases} 4r-3, & \textit{if } r^2 \leq n < r^2 + \lfloor \frac{r}{2} \rfloor \\ 4r-2, & \textit{if } r^2 + \lfloor \frac{r}{2} \rfloor \leq n < r^2 + r \\ 4r-1, & \textit{if } r^2 + r \leq n < r^2 + \lfloor \frac{3r+1}{2} \rfloor \\ 4r, & \textit{if } r^2 + \lfloor \frac{3r+1}{2} \rfloor \leq n < (r+1)^2 \end{cases}$$

Proof. It can be easily verified that

$$\begin{aligned} I(P_2 \times P_1) &= 2 = 4 \times 1 - 2 \\ I(P_2 \times P_2) &= 3 = 4 \times 1 - 1 \\ I(P_2 \times P_3) &= 4 = 4 \times 1. \end{aligned}$$

Let $n \geq 4$ and S be a minimum I-set with smallest number of components of order $m = m(G-S)$. Then by Remark 5, it follows that $I(S) \geq min\{f(t) \mid t \in \mathbb{N}\}$. Hence by Lemma 6, we have

$$I(G) \geq \begin{cases} 4r-3, & \textit{if } r^2 \leq n < r^2 + \lfloor \frac{r}{2} \rfloor \\ 4r-2, & \textit{if } r^2 + \lfloor \frac{r}{2} \rfloor \leq n < r^2 + r \\ 4r-1, & \textit{if } r^2 + r \leq n < r^2 + \lfloor \frac{3r+1}{2} \rfloor \\ 4r, & \textit{if } r^2 + \lfloor \frac{3r+1}{2} \rfloor \leq n < (r+1)^2 \end{cases}$$

We now construct a suitable S with $I(S) = min\{f(t) \mid t \in \mathbb{N}\}$ to prove the reverse inequality. Let

$$A = \{u_{r+1}, u_{2r+1}, u_{3r+2}, u_{4r+2}, \ldots, u_{\ell-r}, u_\ell\}$$

and

$$B = \{v_r, v_{2r+1}, v_{3r+1}, v_{4r+2}, \ldots, v_{\ell-r-1}, v_\ell\}.$$

Case 1. $r^2 \leq n < r^2 + \lfloor \frac{r}{2} \rfloor$. We have two cases.

Case 1a. Let r be even.
In this case, $\lfloor \frac{r}{2} \rfloor = \frac{r}{2}$ and $n = r^2 + \frac{r}{2} - 1 - s, s \geq 0$. If $s \geq 1$, define

$$S = A \cup B \cup \{u_{\ell+ir}, v_{\ell+ir} \mid 1 \leq i \leq 2s - 1\}$$

where $\ell = (\frac{r}{2} - s)(2r + 1)$.
In this case, $|S| = 2r - 2, m(G - S) = 2r - 1$ and $I(S) = 4r - 3$.
If $s = 0$, then $n = r^2 + \frac{r}{2} - 1$. Define

$$S = A \cup B \cup \{u_{\ell+r+1}, v_{\ell+r}\}$$

where $\ell = (\frac{r}{2} - 1)(2r + 1)$.
In this case, $|S| = 2r - 2, m(G - S) = 2r - 1$ and $I(S) = 4r - 3$.

Case 1b. Let r be odd.
In this case, $\lfloor \frac{r}{2} \rfloor = \frac{r-1}{2}$ and $n = r^2 + \frac{r-1}{2} - 1 - s, s \geq 0$. For $s \geq 1$, define

$$S = A \cup B \cup \{u_{\ell+ir}, v_{\ell+ir} \mid 1 \leq i \leq 2s\},$$

where $\ell = (\frac{r-1}{2} - s)(2r + 1)$.
For $s = 0$, define

$$S = A \cup B$$

where $\ell = (\frac{r-1}{2})(2r + 1)$. In each case, $|S| = 2r - 2, m(G - S) = 2r - 1$ and $I(S) = 4r - 3$.

Case 2. $r^2 + \lfloor \frac{r}{2} \rfloor \leq n < r^2 + r$.
In this case, $n = r^2 + r - 1 - s,\ s \geq 0$. Define S as follows:

$$S = \{u_{r+1}, v_{r+1}, u_{2r+2}, v_{2r+2}, \ldots, u_{r^2-1}, v_{r^2-1}\}$$

It is easy to see that $|S| = 2r - 2, m(G - S) = 2r$ and $I(S) = 4r - 2$.

Case 3. $r^2 + r \leq n < r^2 + \lfloor \frac{3r+1}{2} \rfloor$. We make two cases:

Case 3a. Suppose r is even.
In this case, $\lfloor \frac{3r+1}{2} \rfloor = \frac{3r}{2}$ and $n = r^2 + r + \frac{r}{2} - 1 - s, s \geq 0$. Define

$$S = A \cup B \cup \{u_{\ell+ir}, v_{\ell+ir} \mid 1 \leq i \leq 2s\},$$

where $\ell = (\frac{r}{2} - s)(2r + 1)$. In this case, $| S | = 2r, m(G - S) = 2r - 1$ and $I(S) = 4r - 1$.

Case 3b. Suppose r is odd.
In this case, $\lfloor \frac{3r+1}{2} \rfloor = \frac{3r+1}{2}$ and $n = r^2 + r + \frac{r+1}{2} - 1 - s, s \geq 0$. If $s \geq 1$, define:

$$S = A \cup B \cup \{u_{\ell+ir}, v_{\ell+ir} \mid 1 \leq i \leq 2s - 1\},$$

where $\ell = (\frac{r+1}{2} - s)(2r + 1)$. If $s = 0$, then define:

$$S = A \cup B,$$

where $\ell = (\frac{r+1}{2})(2r + 1)$. In each case, $| S | = 2r, m(G - S) = 2r - 1$ and $I(S) = 4r - 1$.

Case 4. $r^2 + \lfloor \frac{3r+1}{2} \rfloor \leq n < (r + 1)^2$.
In this case, we define S as follows:

$$S = \{u_{r+1}, v_{r+1}, u_{2r+2}, v_{2r+2}, \ldots, u_{r^2-1}, v_{r^2-1} u_{r^2+r}, v_{r^2+r}\}$$

Clearly, $| S | = 2r, m(G - S) = 2r$ and $I(S) = 4r$.
Hence the theorem follows. □

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A Note on Cordial Labelings of Multiple Shells

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Let G be a graph with vertex set V and edge set E. A vertex labelling $f : V \longrightarrow \{0, 1\}$ induces an edge labelling $\overline{f} : E \longrightarrow \{0, 1\}$ defined by $\overline{f}(uv) = | f(u) - f(v) |$. Let $v_f(0), v_f(1)$ denote the number of vertices v with $f(v) = 0$ and $f(v) = 1$ respectively. Let $e_f(0), e_f(1)$ be similarly defined. A graph is said to be **cordial** if there exists a vertex labeling f such that $| v_f(0) - v_f(1) | \leq 1$ and $| e_f(0) - e_f(1) | \leq 1$. In this paper, we show that every multiple shell $MS\{n_1^{t_1}, \ldots, n_r^{t_r}\}$ is cordial for all positive integers $n_1, \ldots, n_r, t_1, \ldots, t_r$.

Introduction

Throughout this paper, all graphs are finite, simple and undirected. Let $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G. A mapping $f : V(G) \longrightarrow \{0, 1\}$ is called a binary vertex labeling of G and $f(v)$ is called the label of the vertex v under f. For an edge $e = uv$, the induced edge labeling $\overline{f} : E(G) \longrightarrow \{0.1\}$ is given by $\overline{f}(e) = |f(u) - f(v)|$. Let $v_f(0), v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and let $e_f(0), e_f(1)$ be the number of edges having labels 0 and 1 respectively under $\overline{f}$.

Definition. A binary vertex labeling of a graph G is called a **cordial labeling** if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph G is called **cordial** if it admits a cordial labeling.

Cordial graphs were first introduced by Cahit as a weaker version of both graceful and harmonious graphs.[1]

Definition. A shell S_n of width n is a graph obtained by taking $n - 3$ concurrent chords in a cycle C_n on n vertices. The vertex at which all the chords are concurrent is called the **apex**. The two vertices on C_n adjacent to the apex have degree 2, the apex has degree $n - 1$ and all the other vertices have degree 3 each.

The shell S_n, also called a fan f_{n-1}, was proved to be cordial for all $n \geq 4$.[1]

Definition. A multiple shell $MS\{n_1^{t_1}, \ldots, n_r^{t_r}\}$ is a graph formed by t_i shells of width n_i each, $1 \leq i \leq r$, which have a common apex.

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Thus a multiple shell is a one point union of many shells. If there are in all k shells with a common apex, then it is called a k-tuple shell. If $k = 2$ we call such a multiple shell a double shell. A multiple shell is said to be balanced if it is of the form $MS\{p^t\}$ or of the form $MS\{p^t, (p+1)^s\}$. The balanced multiple shell $MS\{n^t\}$ was proved to be cordial by Sze-Chin Shee and Yong Song Ho.[2]

In this note, we show that all multiple shells are cordial.

Cordial Labelings

We begin by listing first, some binary labelings of the shell $S_n, n \geq 3$, which will be used to construct a cordial labeling of multiple shells. Let $V(S_n) = \{u, v_1, \dots, v_{n-1}\}$ where u is the apex and let $E(S_n) = \{u\, v_i \mid 1 \leq i \leq n-1\} \bigcup \{v_i\, v_{i+1} \mid 1 \leq i \leq n-2\}$.

Clearly $\mid V(S_n) \mid = n$ and $|E(S_n)| = 2n - 3$. Let g_1, g_2, g_3, g_4, g_5 be binary labelings of S_n defined as follows:

(i) $g_1(u) = 0, g_1(v_i) = 1$, if $i \equiv 1, 2 \bmod 4$ and $g_1(v_i) = 0$, if $i \equiv 0, 3 \bmod 4$. (Cordial for all n)

(ii) $g_2(u) = 0, g_2(v_i) = 0$, if $i \equiv 1, 2 \bmod 4$ and $g_2(v_i) = 1$, if $i \equiv 0, 3 \bmod 4$. (Cordial iff $n \equiv 1 \bmod 4$.)

(iii) $g_3(u) = 0, g_3(v_i) = 1$, if $i \equiv 1, 2 \bmod 4$ and $g_3(v_i) = 0$, if $i \equiv 0, 3 \bmod 4$, $i \neq n-3, n-1$. Let $g_3(v_{n-3}) = 0$ and $g_3(v_{n-1}) = 1$. (Cordial for all n)

(iv) $g_4(u) = 0, g_4(v_i) = 1$, if $i \equiv 1, 2 \bmod 4$ and $g_4(v_i) = 0$, if $i \equiv 0, 3 \bmod 4, i \neq n-2$. Let $g_4(v_{n-2}) = 0$. (Cordial iff $n \equiv 1, 2, 3 \bmod 4$.)

(v) $g_5(u) = 0, g_5(v) = 1$, if $i \equiv 1, 2 \bmod 4$ and $g_5(v_i) = 0$, if $i \equiv 0, 3 \bmod 4, i \neq n-1$. Let $g_5(v_{n-1}) = 0$. (Cordial iff $n \equiv 0, 1, 3 \bmod 4$.)

Theorem 1. *All double shells $MS\{n_1, n_2\}$ are cordial for all $n_1, n_2 \geq 4$.*

Proof. Consider the double shell $MS\{n_1, n_2\}$ with $n_1 + n_2 - 1$ vertices and $2(n_1 + n_2) - 6$ edges, which is the one point union of two shells S_{n_1} and S_{n_2} with common apex u.

Let h be a binary labeling defined on the vertex set of $MS\{n_1, n_2\}$ by $h(v) = f(v)$, if $v \in V(S_{n_1})$ and $h(v) = g(v)$, if $v \in V(S_{n_2})$, where f, g are binary labelings of S_{n_1} and S_{n_2} chosen suitably from amongst the binary labelings g_1, g_2, g_3, g_4, g_5 described before. Irrespective of the chioces of f, g we have $v_h(0) = v_f(0) + v_g(0) - 1$, $v_h(1) = v_f(1) + v_g(1)$, $e_h(0) = e_f(0) + e_g(0)$ and $e_h(1) = e_f(1) + e_g(1)$. We choose f, g as follows:

Let $n_1 \equiv r \bmod 4$, $n_2 \equiv s \bmod 4$ where $r, s = 0, 1, 2, 3$. Suppose $r \leq s$.

Case 1. Both r, s are even. Then choose $f = g_1$ and $g = g_2$.

Case 2. Both r, s are equal to 1. Then choose $f = g_1$ and $g = g_3$.

Case 3. $r = 1, s = 3$. Then choose $f = g_1$ and $g = g_1$.

Case 4. $r = 0, s = 1$ OR $r = 1, s = 2$. Then choose $f = g_1$ and $g = g_1$.

Case 5. $r = 0, s = 3$ OR $r = 2, s = 3$. Then choose $f = g_1$ and $g = g_4$.

Case 6. $r = 3, s = 3$. Then choose $f = g_5$ and $g = g_4$.

One can easily see that in each of these cases, $e_h(0) = e_h(1) = n_1 + n_2 - 3$. The following table shows that the binary labeling defined here is cordial.

r	s	f	g	$v_h(0)$	$v_h(1)$
0	0	g_1	g_2	$(n_1+n_2)/2$	$(n_1+n_2)/2-1$
0	1	g_1	g_1	$(n_1+n_2-1)/2$	$(n_1+n_2-1)/2$
0	2	g_1	g_2	$(n_1+n_2)/2$	$(n_1+n_2)/2-1$
0	3	g_1	g_4	$(n_1+n_2-1)/2$	$(n_1+n_2-1)/2$
1	1	g_1	g_3	$(n_1+n_2)/2$	$(n_1+n_2)/2-1$
1	2	g_1	g_1	$(n_1+n_2-1)/2$	$(n_1+n_2-1)/2$
1	3	g_1	g_1	$(n_1+n_2)/2$	$(n_1+n_2)/2-1$
2	2	g_1	g_2	$(n_1+n_2)/2$	$(n_1+n_2)/2-1$
2	3	g_1	g_4	$(n_1+n_2-1)/2$	$(n_1+n_2-1)/2$
3	3	g_5	g_4	$(n_1+n_2)/2$	$(n_1+n_2)/2-1$

□

Now, repeatedly using double shells, we prove the following:

Theorem 2. *All multiple shells are cordial.*

Proof. Consider a multiple shell $MS\{n_1^{t_1}, \ldots, n_r^{t_r}\}$. Let α be the number of shells with even width and let β be the number of shells of odd width. Clearly $\alpha + \beta = t_1 + \cdots + t_r$.

Firstly, form pairs of the shells of even width. For each such resulting double shell, label the vertices using the cordial labeling of Case 1, Theorem 1. At the end of this step either no shell or one shell of even width will remain.

Now, pair the shells of odd width. For each such pair, use the cordial labeling of Case 2, Case 3 or Case 6 of Theorem 1 as needed. Again at the end, either no shell or one shell of odd width will remain.

If no shell is remaining, the labeling obtained will be cordial. If only one shell remains of even or odd width, use labeling g_1 for this shell. On the other hand if one shell of even width and one shell of odd width remain, use the cordial labeling of Case 4 or of Case 5 of Theorem 1. One can easily see that the resulting labeling is cordial. □

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A Report on Additive Complements of the Squares

D.S. Ramana

1. Introduction

This article provides an account of some recent investigations into the behaviour of additive complements of the sequence of squares. We begin by defining this notion.

Definition 1. Given an integer $N \geq 1$, we call a subset B of the integers $\{0, 1, \ldots, N\}$ an *additive complement of the squares upto N* if it is such that every integer n satisfying $0 \leq n \leq N$ can be expressed as $n = b + k^2$, for some b in B and some integer k.

Since every integer n satisfying $0 \leq n \leq N$ lies between two consecutive squares not exceeding N, it follows that every such integer n can be expressed as $n = b + k^2$, for some b in the set of integers in the interval $[0, 2\sqrt{N} + 1]$ and some integer k. In other words, the set of integers in the interval $[0, 2\sqrt{N} + 1]$ is an additive complement of the squares upto N, for every integer $N \geq 1$.

For an integer $N \geq 1$, we call an additive complement of the squares upto N *minimal* if there is no additive complement of the squares upto N of strictly smaller cardinality; we let $b(N)$ denote the cardinality of a *minimal* additive complement of the squares upto N. From the example mentioned in the preceding paragraph it follows that $2\sqrt{N} + 2 \geq b(N)$. Combining this remark with the simple observation that $b(N)\sqrt{N} \geq N$, we obtain the inequalities $2\sqrt{N} + 2 \geq b(N) \geq \sqrt{N}$. If we now define α to be the *liminf* of the sequence $b(N)/\sqrt{N}$, as N tends to infinity, it follows that α lies in the closed interval $[1, 2]$.

The *main problem* on additive complements of the squares is to determine the value of α. This problem, which is open at present, owes its birth to a question of P. Erdös [8], who asked if one may improve the trivial inequality $\alpha \geq 1$ to $\alpha > 1$. This latter question was answered by L. Moser [10], who proved $\alpha > 1.06$. Subsequently, a number of authors worked on the question of improving this lower bound (see [1], [2], [3], [5], [6], [7], [9]). The best known result on this problem is

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the following theorem, which is independently due to J. Cilleruelo [6], L. Habsieger [9], R. Balasubramanian and D.S. Ramana (*unpublished*).

Theorem 1. *α lies in the closed interval* $[4/\pi, 2]$.

In Section 3 we will outline a proof of Theorem 1 based on a refinement of the method in [3]. Before doing this, however, we shall describe a method of Moser [10], with which we will contrast our method.

2. Moser's Method

Suppose that B is an additive complement of the squares upto an integer N and that the cardinality of B is $b(N)$. Moser's method starts with the following obvious relation between arithmetic means.

$$\frac{1}{b(N)}\sum_{b\in B} b + \frac{1}{[\sqrt{N}]}\sum_{1\leq k^2\leq N} k^2 = \frac{1}{[\sqrt{N}]b(N)}\sum_{1\leq k^2\leq N}\sum_{b\in B} b+k^2 \tag{1}$$

Since B is an additive complement of the squares upto N, the double sum on the right hand side of (1) is bounded below $N^2/2$. The second term on the left hand side of (1) is $\backsim N/3 + O(\sqrt{N})$. Combining these two remarks with (1) we obtain the following lower bound for the average value of the elements of B.

$$\frac{1}{b(N)}\sum_{b\in B} b \geq \frac{N^{3/2}}{2b(N)} - \frac{N}{3} + O(\sqrt{N}) \tag{2}$$

The crux of Moser method is to now combine (2) with an upper bound for average value of the elements of B based on a simple estimate for the number of integers *outside* the interval $[1, N]$ that are expressible as $b + k^2$. Indeed, for any b in B and k such that $N - b < k^2 \leq N$, the numbers $b + k^2$ are clearly $> N$. Hence, for any b in B at least $[\sqrt{N}] - [\sqrt{N-b}]$ of the integers of the form $b + k^2$ are $> N$. From this remark we have (3).

$$\begin{aligned}\sqrt{N}b(N) &\geq N + \sum_{b\in B}([\sqrt{N}] - [\sqrt{N-b}]) \geq N \\ &+ \sum_{b\in B}(\sqrt{N} - \sqrt{N-b}) + O(b(N))\end{aligned} \tag{3}$$

Applying the mean value theorem to the function $\sqrt{t}$ on the interval $(N - b, N)$ we obtain $\sqrt{N} - \sqrt{N-b} \geq b/2\sqrt{N}$. Combining this inequality with (3) and dividing throughout by $b(N)/\sqrt{N}$, we have (4).

$$2N \geq \frac{2N^{3/2}}{b(N)} + \frac{1}{b(N)}\sum_{b\in B} b + O(\sqrt{N}) \tag{4}$$

Combining (2) and (4) and rearranging terms we obtain (5).

$$\frac{7N}{3} \geq \frac{5N^{3/2}}{2b(N)} + O(\sqrt{N}) \tag{5}$$

Dividing throughout by N and taking *liminf* on both sides as N tends to infinity, we obtain $\alpha \geq 15/14 > 1.07$ (in fact, Moser is somewhat lax with his calculations here and so obtains only $\alpha > 1.06$).

Moser's idea of considering the contribution due to integers outside the interval $[1, N]$ that are expressible as $b + k^2$ was applied by R. Donagi and M. Herzog in [7] to show $\alpha \geq 1.124$. This was followed by H.L. Abott, [1], who applied (3) along with the analogous inequality for the *sum* of the integers outside $[0, N]$ that are expressible as $b + k^2$ to obtain $\alpha \geq 1.147$ (in addition, Abott's proof uses other substantial innovations that we do not elaborate upon here).

3. Outline of the Proof of Theorem 1

In contrast to Moser's method, our method exploits the multiplicity, *on the average*, with which integers *inside* the interval $[0, N]$ are expressible as $b + k^2$. This we do by means of Lemma 1 below, which is immediate from the definition of additive complements (Definition 1). This lemma was introduced by R. Balasubramanian [2] in order to deduce $\alpha \geq 1.154$. It was subsequently used by R. Balasubramanian and K. Soundararajan [5] to show $\alpha \geq 1.245$. This lemma is at the base of [6], [9] as well.

Lemma 1. *When N is an integer* ≥ 1, *B is an additive complement of the squares upto N and* $f(t)$ *is a real valued function which is* ≥ 0 *for all* $t \geq 0$, *one has the following relation.*

$$\sum_{0 \leq b+k^2 \leq N} f(b+k^2) = \sum_{b \in B} \sum_{0 \leq k \leq \sqrt{N-b}} f(b+k^2) \geq \sum_{1 \leq n \leq N} f(n) \tag{6}$$

In effect, Theorem 1 is proved by exploiting the inequality (6) for the functions $f_m(t) = t^m$, for integers $m \geq 1$ and finally letting m tend to infinity. We give here the main steps in the proof of Theorem 1, refering the reader to [11], Chapter 2 for the details.

Let B be an additive complement of the squares upto N. Let $B(N)$ be the cardinality of the set B. For any t in $[0, 1]$, let $\beta(Nt)$ denote the number of b in B that are $< Nt$.

When m is an integer ≥ 1 and t is in $[0, 1]$, we define the function $\phi_m(t)$ by the following relation and write $g_m(t)$ to denote $-\phi'_m(t)$.

$$\phi_m(t) = \int_0^{1-t} \frac{(u+t)^m}{2\sqrt{u}} du \tag{7}$$

Applying (6) to the functions $f_m(t) = t^m$, for integers $m \geq 1$, and then replacing the summations in (6) by corresponding integrals and finally integrating by parts, we obtain the following relation.

$$\frac{B(N)}{N} + \int_0^1 \frac{\beta(Nt)}{\sqrt{N}} g_m(t)\, dt \geq \frac{1}{m+1} \tag{8}$$

We then verify the following properties of the "kernels" $g_m(t)$.

Lemma 2. *When m is an integer* ≥ 1, *we have the following.*

(i) $g_m(t)$ *has a unique zero* x_m *in the interval* $[0, 1]$. *Further,* $g_m(t)$ *is* < 0 *on* $[0, x_m)$ *and is* > 0 *on* $(x_m, 1]$.
(ii) $\lim_{m\to\infty}(m+1)\int_{x_m}^1 (1-\sqrt{t})g_m(t)\, dt = 0$.
(iii) $\int_0^1 \sqrt{t} g_m(t)\, dt = \pi/4(m+1)$.

Now we let N_k be a sequence of integers such that $b(N_k)/\sqrt{N}_k \to \alpha$ as $k \to \infty$. We apply (8) to the N_k and *minimal* additive complements B_k of the squares upto N_k. Taking $\limsup_{k\to\infty}$ and noting $\limsup_{k\to\infty} b(N_k)/N_k$ is 0, we obtain (9), for all integers $m \geq 1$.

$$\limsup_{k\to\infty} \int_0^{x_m} \frac{\beta(N_k t)}{\sqrt{N_k t}} \sqrt{t} g_m(t)\, dt + \limsup_{k\to\infty} \int_{x_m}^1 \frac{\beta(N_k t)}{\sqrt{N_k}} g_m(t)\, dt \geq \frac{1}{m+1} \tag{9}$$

By (i) of Lemma 2, $g_m(t)$ is < 0 on $[0, x_m)$. It follows by an application of Fatou's Lemma (see [12], Sec. 1.28, page 23) that the first term on the left hand side of (9) is $\leq \alpha \int_0^{x_m} \sqrt{t} g_m(t)\, dt$. Since $g_m(t)$ is > 0 on $(x_m, 1]$ it follows from the choice of N_k that the second term on the left hand side of (9) is $\leq \alpha \int_{x_m}^1 g_m(t)\, dt$. Combining these upper bounds with (9) we obtain (10) for each integer $m \geq 1$.

$$\alpha \int_0^1 \sqrt{t} g_m(t)\, dt + \alpha \int_{x_m}^1 (1-\sqrt{t}) g_m(t)\, dt \geq \frac{1}{m+1} \tag{10}$$

Multiplying both sides of (10) by $m + 1$ and taking into account (ii) and (iii) of Lemma 2 while letting $m \to \infty$, we obtain $\alpha \geq 4/\pi$. Since we have already noted in Section 1 that $2 \geq \alpha$, this completes the proof of Theorem 1.

4. An Intuitive Explanation

With the intention of providing an intuitive explanation for why our method gives the lower bound $4/\pi$ for α, we consider the following situation. Let B be an *infinite* sequence of nonnegative integers and, for any $t > 0$, let $\beta(t)$ be the number of b in B that are $< t$. Further, for an integer $N \geq 1$ and for each integer n satisfying $1 \leq n \leq N$, let $r(N, n)$ be the number of ordered pairs of integers (b, k) such that

$n = b + k^2$ and finally let $R(N)$ denote the average $\sum_{1 \le n \le N} r(N,n)/N$. With this notation we have the following relation.

$$\sum_{1 \le n \le N} r(N,n) = \sum_{b \in B} \sum_{1 \le k \le \sqrt{N-b}} 1 = \sum_{b \in B} \sqrt{N-b} + O(\beta(N)) \tag{11}$$

Writing the last term in (11) as an integral with respect to $\beta(Nt)$ and integrating by parts we have (12).

$$\sum_{b \in B} \sqrt{N-b} + O(\beta(N)) = N^{1/2} \int_0^1 \frac{\beta(Nt)\,dt}{2\sqrt{1-t}} + O(\beta(N)) \tag{12}$$

Suppose that $\beta(t) \sim a\sqrt{t}$, for a real number $a > 0$. If we now require that the set B, when added to the squares, represent all integers *on the average* (in other words, if we require that $\liminf_{N\to\infty} R(N) \ge 1$), then by (11), (12) and the dominated convergence theorem we must have (13).

$$\begin{aligned} a\int_0^1 \frac{\sqrt{t}}{2\sqrt{1-t}}dt &= \liminf_{N\to\infty}\left(\int_0^1 \frac{\beta(Nt)}{\sqrt{Nt}}\frac{\sqrt{t}}{2\sqrt{1-t}}dt + O\left(\frac{\beta(N)}{N}\right)\right) \\ &= \liminf_{N\to\infty} R(N) \ge 1 \end{aligned} \tag{13}$$

Since the integral $\int_0^1 \frac{\sqrt{t}}{2\sqrt{1-t}}\,dt$ is equal to $\pi/4$, we must then have $a \ge 4/\pi$.

The circumstances leading to the lower bound $\alpha \ge 4/\pi$ in Theorem 1 are similar to the situation just described. In effect, applying the inequality (6) only to the *monotonic* functions $f_m(t) = t^m$ restricts us from exploiting anything more than the fact that the additive complements of the squares upto an integer N represent all integers not exceeding N *on the average* (by an abuse of language, we say that an additive complement of the squares B represents an integer n if $n = b + k^2$, for some b in B and some integer k). Since, in fact, additive complements of the squares upto an integer N represent *all* integers not exceeding N (and not just on the average), we expect that the lower bound for α given by Theorem 1 may be improved.

5. A Possible Direction

The "Greedy Algorithm" lead us to the following hueristics. Since an additive complement represents all integers, it must necessarily contain "small" integers. If, in addtion, it is required to be minimal then it must make do with small integers. These considerations suggest that *there exist minimal additive complements containing only small elements.* Therefore we may ask the following question.

Question 1. *Can the lower bound* $4/\pi$ *for* α *be improved under the assumption that there exists a* δ *in* $(0, 1)$ *such that for all large integers N there is a minimal additive complement of the squares contained in the interval* $[0, \delta N]$?

This question is answered by following theorem due to R. Balasubramanian and D.S. Ramana [4]. This theorem is an improvement over a result obtained (by a different method) by Zhai [13].

Theorem 2. *If, for a δ in the interval* $(0, 1)$ *and all large integers N, there is a minimal additive complement of the squares upto N contained in the interval* $[0, \delta N]$, *then one has the following inequality.*

$$\alpha \geq \frac{2}{\frac{\sqrt{1-\delta}}{(1+\sqrt{\delta})} + \sin^{-1}(\sqrt{\delta})} \tag{14}$$

The right hand side of (14) is a continuous function of δ taking the value 2 when δ is 0 and the value $4/\pi$ when δ is 1. In particular, therefore, it follows from Theorem 2 and the inequality $2 \geq \alpha$, that if, for all large N, there exists a minimal additive complement of the squares upto N all of whose elements are $o(N)$, then $\alpha = 2$.

Theorem 2 and the general belief that α must be 2 (see [9], [6]) suggest the following question.

Question 2. *Does there exist a δ in the open interval* $(0, 1)$ *with the following property? For all large enough N, there exists a minimal additive complement B of the squares upto N such that the number of elements of B that are* $> \delta N$ is $o(N^{1/2})$.

We are unaware of any result in the literature in the direction of Question 2.

Finally, we note that theorems analogous to Theorems 1 and 2 can be proved by a suitable adaptation of our methods in the general case of additive complements of higher powers. We refer the reader to [11], Chapter 2 for such results (see also [7], [6], [9]).

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Transcendental Infinite Sums and Some Related Questions

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Erdős and Chowla put forward some questions regarding non-vanishing of certain infinite sums. In this article, we present an expository account of results obtained in that direction. These include some interesting results of Baker, Birch and Wirsing and some recent work of the present author jointly with Saradha, Shorey and Tijdeman.

1. Introduction

A crucial step in the known proofs of Dirichlet's famous theorem about the existence of infinitely many primes in the sequence $\{qn+a\}_{n=1}^{\infty}$, where $(q, a) = 1$, is the fact that for a non-principal character χ mod q,

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n} \neq 0,$$

the series being convergent.

In general, for a periodic function f, questions about deciding when$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0$, provided the series is convergent, are not only interesting on their own but also lead to important results.

We discuss about some research in this direction, initially motivated by some questions put forward by Erdős and Chowla.

2. Early Results

We start our account with an old conjecture of Erdős (see Livingston [8]):

Conjecture 1. If $q > 1$ is a positive integer and f is a number-theoretic function with period q for which $f(n) \in \{-1, 1\}$ when $n = 1, 2, \ldots, q-1$ and $f(q) = 0$, then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0$$

whenever the series is convergent.

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Livingston [8] showed that Conjecture 1 is true provided certain set of numbers is linearly independent over the algebraic numbers. More precisely he proves that Conjecture 1 is true if the numbers

$$\pi, \log\left(2\sin\frac{\pi}{q}\right), \log\left(2\sin\frac{2\pi}{q}\right), \ldots, \log\left(2\sin\frac{(q-1)\pi}{q}\right)$$

are linearly independent over the algebraic numbers, when q is odd. He gives a similar criterion when q is even.

We shall, in due course, come to further results due to Baker, Birch and Wirsing, Okada and Tijdeman towards Conjecture 1.

Before that, incorrectly attributing to Erdős, in an address delivered before the annual meeting of the American Mathematical Society in 1949 Chowla [5] had stated the following conjecture.

Conjecture 2. Let $q > 1$ be a positive integer and f a number-theoretic function with period q such that $f(n) \neq 0$ for some positive integer n. Then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0$$

whenever the series is convergent.

Observing that Conjecture 2 is trivially true for $q = 2$ (Since in that case, the function f has to be a multiple of $(-1)^{n+1}$, the sum being therefore a multiple of $\log 2$.), Livingston [8] exhibited counter examples to Conjecture 2 for $q > 2$. To be more precise, Livingston showed that for the result to be true, f can not be allowed to assume arbitrary real or complex values. We shall have a look at this and some other results of Livingston at the end of this section. Simpler counter examples (where f takes only integer values) from classical literature have been given later in Tijdeman [10] (see also [1]). For example, the following infinite sum is zero:

$$1 - \frac{3}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{3}{6} + \frac{1}{7} + \frac{1}{8} + \ldots.$$

Since experts feel that for Chowla and Erdős, a number-theoretic function meant an integral valued function, the counter example due to Tijdeman is more to the point.

Meanwhile, Chowla himself must have realised about the necessity of having some more conditions on the function f in Conjecture 2. Whereas (in written communication with Erdős) Livingston [8] had obtained the version of Erdős' conjecture as stated in Conjecture 1, few years after the appearance of the above mentioned paper of Livingston [8], in a conference in 1969 (see [3]), Chowla raised the following modified version of the question.

Question 1. *Does there exist a rational-valued function $f(n)$, periodic with prime period p, such that*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = 0. \tag{1}$$

Long before Chowla (see [6]) had shown that this does not hold for odd functions f if in addition $(p-1)/2$ happens to be a prime. This additional condition was subsequently removed by Siegel (see [6] and [3]). Later, (see [3]), Chowla showed that the result is true for even functions f provided only that $f(0) = 0$.

In 1973, Baker, Birch and Wirsing [3] proved a result which settled the above mentioned question (Question 1) of Chowla. In fact, they proved more. We shall discuss about their result in the next section.

Before we end this section, we shall like to discuss about some interesting results obtained by Livingston [8]. We start with the statement of the following result.

Theorem 1. (Livingston) *Let $\{a_n\}_{n=1}^{\infty}$ be a real or complex valued sequence and q a positive integer greater than 2. If $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=0}^{\infty}(a_{nq+r} - a_{nq+r+1})$ is convergent for $r = 1, 2, \ldots, q-1$, then there is a non-zero sequence $\{b_n\}_{n=1}^{\infty}$ for which $b_{n+q} = b_n$ for $n = 1, 2, 3, \ldots$, $\sum_{n=1}^{q} b_n = 0$ and $\sum_{n=1}^{\infty} a_n b_n = 0$.*

With $a_n = 1/n$ and $f(n) = b_n$, Theorem 1 provides us with counter examples for Conjecture 2 for $q > 2$. Livingston also showed that the additional condition $f(q-n) = -f(n)$ (as had been thought to be in [5]) is also not enough for Conjecture 2 to be true for general q. More precisely, it has been proved to be false for $q > 4$. As has been pointed out in [8], when $q = 2$, the only number theoretic function f satisfying $f(2+n) = f(n)$ and $f(2-n) = -f(n)$ is the zero function whereas for $q = 3$ and 4, the only functions under consideration are multiples of $\sin(2n\pi/3)$ and $\sin(n\pi/2)$ respectively and Conjecture 2 is true.

We conclude this section with the following interesting result corresponding to multiplicative functions, due to Livingston [8]; further results due to Tijdeman [10] for multiplicative and completely multiplicative functions will be described later on.

Theorem 2. (Livingston) *Let p be a prime and f a real or complex valued multiplicative number theoretic function of period p. If $f(n) \neq 0$ for some positive integer n, then*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0$$

whenever the series is convergent.

Proof. First we claim that if a and b are positive integers such that $(a, p) = 1$, then $f(ab) = f(a)f(b)$. To establish the claim, we observe that by Dirichlet's theorem $\{pn+a\}_{n=1}^{\infty}$ contains arbitrarily large primes and hence $(pn+a, b) = 1$, for some positive integer n. Therefore,

$$f(ab) = f((pn+a)b) = f(pn+a)f(b) = f(a)f(b).$$

If $f(p) \neq 0$, then $(a, p) = 1$ would imply $f(a) = 1$. The convergence of $\sum_{n=1}^{\infty} \frac{f(n)}{n}$ would imply $\sum_{k=1}^{p} f(k) = 0$. Since $f(k) = 1$ for $k = 1, 2, \ldots, p-1$, we have $f(p) = 1 - p$. Therefore, in this case

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = \sum_{k=0}^{\infty} \left(\frac{1}{pk+1} + \frac{1}{pk+2} + \cdots + \frac{1}{pk+p-1} - \frac{p-1}{pk+p} \right) > 0.$$

On the other hand, if $f(p) = 0$, then observing that $f(1) \neq 0$ because of multiplicativity, f is a character modulo p. Also f is non-principal as $\sum_{k=1}^{p} f(k) = 0$. Therefore, Dirichlet's result gives us the non-vanishing of $\sum_{n=1}^{\infty} \frac{f(n)}{n}$ in this case. □

3. The Result of Baker, Birch and Wirsing and Further Results

Following is the statement of the theorem of Baker, Birch and Wirsing [3] which we mentioned in the previous section.

Theorem 3. (Baker, Birch and Wirsing) *If f is a non-vanishing function defined on the integers with algebraic values and period q such that* (i) $f(r) = 0$ *if* $1 < (r, q) < q$, (ii) Φ_q *is irreducible over* $\mathbf{Q}(f(1), \ldots, f(q))$, *where q is any natural number, Φ_q the q-th cyclotomic polynomial and $\mathbf{Q}$ the field of rationals, then*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0.$$

Remark 1. We remark that if q is a prime then (i) is vacuous while (ii) holds trivially if f is rational valued. Thus, Theorem 3 provides an answer in the negative to Question 1.

However, in the general case, both the conditions are necessary. If p is a prime and $q = p^2$, then $f(n)$ defined by

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = (1 - p^{1-s})^2 \zeta(s),$$

shows the necessity of (i).

Again, if χ and χ' are quadratic characters modulo 12 with conductors 3 and 4 respectively, then defining f by $f = 2\chi - \sqrt{3}\chi'$ we see that (ii) is necessary, since $L(1,\chi) = \frac{\pi}{2\sqrt{3}}$ and $L(1,\chi') = \frac{\pi}{3}$.

A Sketch of the Proof of Theorem 3. Here, **Z** will denote the set of integers and **A** will denote the set of algebraic numbers. We shall also use the notations $\xi = \xi_q = e^{2\pi i/q}$. Consider the set

$$\mathcal{F}_q := \{f : \mathbf{Z} \to \mathbf{A} \text{ with period } q \text{ such that (1) holds}\}$$

and let $\mathcal{G}_q$ denote the set of all functions g defined on **Z** which are of the form

$$g(s) = \frac{1}{q}\sum_{r=1}^{q} f(r)\xi_q^{-rs} \quad \text{with } f \in \mathcal{F}_q. \tag{2}$$

Clearly, (2) can be inverted by

$$f(r) = \sum_{s=1}^{q} g(s)\xi_q^{rs}. \tag{3}$$

We observe that $\mathcal{F}_q$ and $\mathcal{G}_q$ are vector spaces of the same dimension over **A** and for $f \in \mathcal{F}_q$,

$$\sum_{r=1}^{q} f(r) = 0. \tag{4}$$

We also observe that (4) implies that $g(0) = 0$ for $g \in \mathcal{G}_q$.

With the help of these observations, it is not difficult to check that if $g \in \mathcal{G}_q$, then

$$\sum_{s=1}^{q-1} g(s)\log(1 - \xi^s) = 0. \tag{5}$$

Conversely, if g is an algebraic-valued periodic function of period q and $g(0) = 0$, then defining f by (3), we see that (5) implies that $g \in \mathcal{G}_q$.

With this characterisation for functions in $\mathcal{G}_q$, and applying Baker's result on linear independence of logarithms, one observes that for any automorphism σ of **A**, $g \in \mathcal{G}_q$ implies that $\sigma g \in \mathcal{G}_q$. With this observation, one obtains the following lemma.

Lemma 1. *If* $f \in \mathcal{F}_q$, σ *is any automorphism of* **A**, *and* h *is an integer defined mod* q *by* $\sigma^{-1}\xi = \xi^h$, *then* $f'(n) := \sigma f(hn)$ *is also in* $\mathcal{F}_q$.

Now we proceed to sketch the proof of Theorem 3.

Let f be as in the statement of Theorem 3. Let χ be any non-principal character mod q and let

$$b(n) = \sum_{h=1}^{q} \chi(h) f(hn). \tag{6}$$

Because of (ii), for any h with $(h, q) = 1$, there exists an automorphism σ of **A** such that $\sigma^{-1}\xi = \xi^h$ and $\sigma f = f$. Therefore, by Lemma 1, for all such h,

$$f(hn) \in \mathcal{F}_q. \tag{7}$$

From (6) and (7), $b \in \mathcal{F}_q$. Also, $b(n) = b(1)\bar{\chi}(n)$ for all n, which implies that $b(1)L(1, \bar{\chi}) = 0$. Since by Dirichlet's theorem, $L(1, \bar{\chi}) \neq 0$, it follows that $b(1) = 0$. Therefore, from definition (6),

$$\sum_{h=1}^{q} \chi(h) f(h) = 0. \tag{8}$$

Again, from (7), one obtains

$$\sum_{n=1}^{\infty} \frac{a(n)}{n} = 0,$$

where

$$a(n) = \sum_{\substack{h=1 \\ (h,q)=1}}^{q} f(hn).$$

On the other hand,

$$\sum_{n=1}^{\infty} \frac{a(n)}{n} = f(0) \cdot D,$$

where

$$D = \sum_{m=0}^{\infty} \left\{ \frac{\phi(q)}{q + mq} - \sum_{\substack{s=1 \\ (s,q)=1}} \frac{1}{s + mq} \right\}.$$

Since D is clearly negative, it follows that $f(0) = 0$ and hence by (4),

$$\sum_{h=1}^{q} \chi_0(h) f(h) = 0 \tag{9}$$

where χ_0 denotes the principal character mod q.

From (8) and (9), $\sum_{h=1}^{q} \chi(h) f(h) = 0$ for all characters χ and hence $f(h) = 0$ for all h with $(h, q) = 1$ and since $f(h) = 0$ for all other h, f vanishes identically.

It was further shown in [3] that if f is a function defined on the integers with algebraic values and period q such that $f(r) = 0$ when $1 < (r, q) < q$ and $\sum_{n=1}^{\infty} \frac{f(n)}{n} = 0$, then f must be odd. A complete description of such functions has also been provided in [3].

Later, in 1982, Okada [9] gave a description of all functions f such that Φ_q is irreducible over $\mathbf{Q}(f(1), \ldots, f(q))$, and $\sum_{n=1}^{\infty} \frac{f(n)}{n} = 0$. From that Okada deduced the following result among other things.

Theorem 4. **(Okada)** *Let $q > 1$ be a positive integer and f a number-theoretic function with period q. If $2\phi(q) + 1 > q$ and $f(n) \in \{-1, 1\}$ when $n = 1, 2, \ldots, q - 1$ and $f(q) = 0$, then*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0$$

whenever the series is convergent.

Coming back to Erdős' conjecture (Conjecture 1), mentioned at the very beginning of Section 2, the theorem (Theorem 3) of Baker, Birch and Wirsing establishes the conjecture when q is prime. The result (Theorem 4) of Okada mentioned above, also solves Erdős' conjecture for certain types of q, for instance, when q is a prime power or the product of two odd primes.

The criteria, from which Okada derives his theorem, were further exploited by Tijdeman [10] to prove several interesting results towards Erdős' conjecture.

The first result of Tijdeman [10], in this direction, says that Erdős' conjecture is true if q is even.

In the case of multiplicative functions, from Okada's results, Tijdeman derives that for some positive integer $q > 1$, if f a number-theoretic function with period q, such that Φ_q is irreducible over $\mathbf{Q}(f(1), \ldots, f(q))$, and $\sum_{n=1}^{\infty} \frac{f(n)}{n} = 0$, then there is a prime divisor p of q with

$$1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \ldots = 0. \tag{10}$$

If it is further assumed that for every prime divisor p of q and any positive integer k, $|f(p^k)| \leq p - 1$, then (10) would imply that for a prime divisor p of q, $f(p^k) = -(p - 1)$ for any positive integer k, which would contradict some other part of the Okada criteria. Thus, Erdős' conjecture is true for multiplicative functions.

Another result of Tijdeman is as follows. If f is completely multiplicative, then from (10) it is deduced that for some prime divisor p of q, $f(p^k) = (-p)^k$ for every positive integer k, which is impossible for a periodic function.

Thus one obtains,

Theorem 5. *For some positive integer $q > 1$, if f a rational valued number-theoretic function with period q, such that f is completely multiplicative, then*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0$$

whenever the series is convergent.

Using Baker's theory, some work done recently in this direction jointly with Saradha, Shorey and Tijdeman [1] gives transcendence of certain infinite sums. For instance, the following theorem has been proved in [1].

Theorem 6. *If f is an algebraic-valued number-theoretic function with period q such that $\sum_{n=1}^{\infty}\frac{f(n)}{n}$ converges, then the sum $\sum_{n=1}^{\infty}\frac{f(n)}{n}$ is either zero or transcendental.*

Combining Theorem 6 with Dirichlet's non-vanishing result, it follows that for an integer $q > 1$, if χ is a non-principal character mod q, then $L(1, \chi)$ is transcendental. From the class number formula for quadratic fields, it is well known that in the particular case $\chi(n) = (\frac{d}{n})$ where d is the discriminant of the quadratic field and $(\frac{d}{n})$ is the Kronecker symbol, the sum

$$\sum_{n=1}^{\infty}\left(\frac{d}{n}\right)\frac{1}{n}$$

is transcendental.

Theorem 6, while combined with Theorem 5 of Tijdeman would imply that for a rational valued completely multiplicative arithmetical function, with period q, $\sum_{n=1}^{\infty}\frac{f(n)}{n}$ is transcendental, whenever the series is convergent.

A Sketch of the Proof of Theorem 6. First we apply a method of Lehmer [7] for expressing certain infinite sums as linear forms in logarithms of algebraic numbers with algebraic coefficients.

More precisely we prove the following result.

Let q be a positive integer. Let $c_1, \ldots, c_m$ be complex numbers and $k_1, \ldots, k_m$, $r_1, \ldots, r_m$ be integers with $0 \le r_\mu < k_\mu$ for $\mu = 1, \ldots, m$. If the double sum on the left hand side converges, then

$$(11) \qquad \sum_{n=0}^{\infty}\sum_{\mu=1}^{m}\frac{c_\mu}{k_\mu n + r_\mu} = \sum_{\mu=1}^{m}\sum_{j=1}^{k_\mu - 1}\frac{c_\mu}{k_\mu}(1 - \zeta_\mu^{-jr_\mu})\log(1 - \zeta_\mu^{j})$$

where ζ_μ is some primitive k_μ-th root of unity for $\mu = 1, \ldots, m$.

By applying (11) with $m = k_\mu = q, r_\mu = \mu, c_\mu = f(\mu)$ for $\mu = 0, \ldots, q-1$, we obtain

$$S = \sum_{n=1}^{\infty}\frac{f(n)}{n} = \sum_{j=1}^{q-1}\beta_j \log(1 - \zeta_q^{j})$$

where the algebraic number β_j is given by

$$\beta_j = \sum_{\mu=0}^{q-1}\frac{f(\mu)}{q}(1 - \zeta_q^{-j\mu}) \text{ for } j = 1, \ldots, q-1.$$

Next, the theorem of Baker [2] is employed which says that if $\alpha_1, \ldots, \alpha_n$ are non-zero algebraic numbers and $\beta_1, \ldots, \beta_n$ are algebraic numbers then $\Lambda = \beta_1 \log\alpha_1 + \cdots + \beta_n \log\alpha_n$ is either zero or transcendental.

Therefore, S is either zero or transcendental and by the result of Baker, Birch and Wirsing (Theorem 3) we get our result.

Even though we can not replace the denominator n in the terms of the series in Theorem 6 with an arbitrary higher degree polynomial (for an odd positive integer k whether $\sum_{n=1}^{\infty} n^{-k}$ is algebraic or not is indeed a very difficult question!), by using partial fractions we could deal with the case when the denominator is a polynomial over the rationals having simple rational zeros. The result obtained in [1] in this direction will imply in particular that

$$\sum_{n=1}^{\infty} \frac{1}{(3n+1)(3n+2)(3n+3)}$$

is transcendental.

Similarly, the result obtained in [1] by putting an exponential polynomial in the numerator implies in particular that

$$\sum_{n=1}^{\infty} \frac{F_n}{n2^n},$$

where $\{F_n\}_{n=1}^{\infty}$ is the Fibonacci sequence, is transcendental.

Finally, we shall like to mention a paper [4] of Peter Bundschuh dealing with related questions. Apart from showing that non-zero values of certain power series is transcendental by the application of Baker's result (mentioned during the proof of Theorem 6) on linear forms in logarithms, [4] contains the following result which belongs to the present theme. Writing $\gamma(n)$ to denote the number of expressions $n = a^b$ with positive integral a, b, Bundschuh deduces from the (unproved) Schanuel conjecture that $\sum_{n=2}^{\infty} \gamma(n)n^{-s}$ is transcendental for each even integer $s \geq 4$.

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The Lehmer Problem on the Euler Totient: A Pendora's Box of Unsolvable Problems

*M.V. Subbarao**

1. Introduction

The celebrated seventy year old, innocent looking problem of D.H. Lehmer [5] asking for composite numbers, if any, satisfying the relation $\phi(n)|(n-1)$, where $\phi(n)$ is the Euler totient, is still unsolved. This is easily seen to be equivalent to asking the

Problem 1.1. Given $n > 1$, n odd and $\phi(n)|(n-1)$, is n necessarily a prime?

It has become as hopeless to settle as the problem of the existence of the odd perfect numbers, and the problem of the existence of the unitary multiperfect numbers, equivalently, whether there exist distinct primes $p_1, \ldots, p_r$ and positive integers $a_1, \ldots, a_r$ such that

$$\left(1+\frac{1}{p_1 a_1}\right)\cdots\left(1+\frac{1}{p_r a_r}\right) = \text{an integer } N > 2.$$

For the case $N = 2$, only five solutions are known [17]. The Lehmer problem (1.1) is equivalent to the following.

(1.2) Do there exist distinct odd primes $p_1, \ldots, p_r$ with $r > 1$ such that

$$p_1 p_2 \ldots p_r - 1 = 0 \bmod (p_1 - 1)\ldots(p_r - 1)?$$

This is a special case of the more general problem raised in 1971 by M.V. Subbarao [14]:

Problem 1.3. Is the relation

$$(p_1^{a_1} \ldots p_r^{a_r}) - 1 \equiv 0 \bmod (p_1^{a_1} - 1)\ldots(p_r^{a_r} - 1)$$

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possible for r distinct primes $p_1, \dots, p_r$ where $r > 1$, and for any positive integers $a_1, \dots, a_r$?

This problem was investigated in considerable detail by V. Siva Rama Prasad and the author (see [11] and [15]). While some partial results are obtained for the two problems (1.2) and (1.3) by several authors, a complete solution is nowhere in sight.

It may be remarked that any integer n satisfying the relation stated in (1.1) does behave like a prime in some respects. For example, it satisfies

$$2^{n-1} \equiv 1 (\bmod n),$$

(recall Fermat's little theorem).

See Landau [4], p. 238, Exercise 14. Note also that the left side of the relation in (1.3) gives the value of the unitary totient function $\phi^*(n)$ for $n = p_1^{a_1} \dots p_r^{a_r}$. Thus we have the curious situation that the unitary analogue of the Lehmer problem (1.1) is, at the same time, a generalization of that problem. The two problems (1.2) and (1.3) are unified and studied in [11] in the more general setting of the regular convolutions introduced by Narkiewich [7].

A companion to the Lehmer problem (1.1) was introduced and studied in 1998 by the author [16]. It asks the following seemingly hopeless

Problem 1.4. Is it possible to have $r > 1$ distinct primes $p_1, \dots, p_r$ such that

$$(p_1 + 1) \dots (p_r + 1) - 1 \equiv 0 \bmod (p_1 \dots p_r)?$$

The problem can of course be generalized by replacing the primes by their powers as in (1.3).

Some results for these problems are obtain in [16]. Notice that the two problems raised (1.1) and (1.4) can be combined and extended as follows.

Problem 1.5. Is there any positive integer a and $r > 1$ distinct primes $p_1, \dots, p_r$ so that we have

$$(p_1 + a) \dots (p_r + a) - 1 \equiv 0 \bmod (p_1 + a - 1) \dots (p_r + a - 1)?$$

This problem of the author is dedicated to the author's friend, colleague and distinguished problem enthusiast Murray Klamkin on his 80th birthday and the author offers a hundred Canadian dollars for the first correct and published solution giving the value of such an integer a. The problem is due to appear soon in the Pi, Mu, Varepsilon Journal.

Needless to say that the problem in (1.4) can be generalized by replacing the primes by their powers. Some other analogous problems are mentioned in the last section.

Indeed Lehmar's Problem (1.1) has opened a Pandora's box of interesting and apparently unsolvable problems.

In this paper, we survey some of the advances made on the problems (1.1) to (1.4). Problem (1.5) and its generalization and the problems mentioned in the last section are open for investigation.

2. Notation and Definitions

In all that follows, unless stated otherwise, n stands for an integer > 1 with the canonical factorization

$$n = p_1^{a_1} p_2^{a_2} \dots p_r^{a_r}, \quad p_1 < p_2 < \dots < p_r.$$

The letters $p, q, p_1, \dots, p_r$ denote primes and the a_i's are integers > 0. $\phi(n)$ is the Euler totient, $\mu(n)$ the Möbius function and $\omega(n)$ denotes the numbers of distinct prime factors of n.

$$\sigma_r(n) = \sum_{d|n} d^r; \quad \sigma_1(n) = \sigma(n).$$

$\psi(n)$ is the Dedekind function representing the sum of those divisors of n whose conjugates are square-free, so that

$$\psi(n) = n \prod_{p|n} \left(1 + \frac{1}{p}\right) = \sum_{d\delta=n} \mu^2(d)\delta.$$

$\phi^*(n)$ is the unitary analogue of the Euler totient and has the evaluation

$$\phi^*(n) = \prod_{i=1}^{r} (p_i^{a_i} - 1).$$

Next,

$$\sigma_r^*(n) = \sum_{\substack{d|n \\ (d,n/d)=1}} d = \prod_{i=1}^{r} (p_i^{a_i} + 1)$$

and $\sigma^*(n) = \sigma_1^*(n)$.

Jordan's totient $J_k(n)$ is given by

$$J_k(n) = n^k \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^k}\right).$$

The Schemmel totient $\phi_a(n)$, $k \geq 1$, is given by

$$\phi_a(n) = n \prod_{i=1}^{r} \left(1 - \frac{a}{p_i}\right),$$

provided all the $p_i > a$, and otherwise $\phi_a(n) = 0$.

Consider the equations

$$k\phi(n) = n - 1 \tag{2.1}$$

and

$$k\phi^*(n) = n - 1. \tag{2.2}$$

The sets S_k and S_k^* are defined thus:

(2.3) S_k is defined as the set of all composite n satisfying (2.1) for $k \geq 1$.

(2.4) S_k^* is defined as the set of all composite n with more than one prime divisor satisfying (2.2) Note that S_1 consists of primes only while S_1^* consists of prime powers.

Lehmer's problem (1.1) is solved if S_k is null for all $k > 1$, and the author's problem (1.3) is solved if S_k^* is nul for all $k > 1$.

3. Some Results for S_k and S_k^*

In this section, unless specified otherwise, n denotes a composite number in S_k for some $k > 1$.

Among the first results about the sets S_k, the following are due to Lehmer [5]. He noted the easy fact that:

(3.1) n is odd and square-free;

(3.2) if p and q divide n; then $q \not\equiv 1 \pmod{p}$;

(3.3) $\omega(n) \geq 7$.

In 1944, Fr. Schuh [9] claimed to have showed that

$$\omega(n) \geq 11. \tag{3.4}$$

However, his proof is incorrect, as shown by E. Lieuwens [6], who supplied the correct proof for (3.4). Further, using the important fact that

$$k \equiv 1 (\bmod 3) \quad \text{if} \quad 3|n \tag{3.5}$$

which was in fact due to Schuh [9]. Lieuwens [6] proved that

$$\omega(n) \geq 212 \quad \text{if} \quad 3|n. \tag{3.6}$$

This was improved by Maso Kishore [3] to

$$\omega(n) \geq 13 \tag{3.7}$$

utilizing extensive computational methods. This was further extended by Cohen and Hagis to

$$\omega(n) \geq 14 \tag{3.8}$$

and also

$$\omega(n) \geq 17 \quad \text{if} \quad (n, 15) = 1. \tag{3.9}$$

It may be remarked that (3.9) is an improvement of the result $\omega(n) \geq 13$ for $(n, 15) = 1$ falsely claimed by Lieuwens [6], who overlooked a computational error.

Note that all results proved for S_k^* are valid for S_k, because $S_k \subset S_k^*$ for each k, $k \geq 1$. For an exhaustive study of the sets S_k^* (and thereby of S_k), see Siva Rama Prasad and the author [13], who have proved several important results. They include a new proof of Lehmer's result (3.3) and the result

$$\omega(n) \geq 1850 \text{ if } 3|n \text{ and } n \in S_k^*, \tag{3.10}$$

which is an improvement of (3.6). The proof needed the help of a computer calculation.

Later Prasad and Rangamma [10] have improved (3.10) using a simple argument by showing

$$\omega(n) \geq 5334 \text{ if } 3|n \text{ and } n \in S_k \text{ with } k > 4. \tag{3.11}$$

Peter Hagis Jr. [2] vastly improved (3.11) by utilizing ingenious theoretical arguments and computer calculations by proving that

$$\omega(n) \geq 298848 \text{ if } 3|n \text{ and } n \in S_k^n \tag{3.12}$$

and in this case,

$$n > 101937047.$$

He has some more interesting theorems for S_k such as

(3.13) For n in S_k with $k \geq 2$ and $(15, n) = 1$, we have $\omega(n) \geq 26$, improving earlier results, and

(3.14) If $n \in S_k$ and $k \geq 3$, then

$$\omega(n) \geq 1991 \text{ and } n > 10^{8171}.$$

We next state two theorems for $n \in S_k$ with $\omega(n) = r$ (fixed).

(3.15) (Pomerance [8]). For every $n \in S_k$ with $\omega(n) = r$ we have $n < r^{2^r}$.

(3.16) This has been improved by Prasad and the author [15] who showed that such an n satisfies

$$n < 2^{2^{r-1}}. \tag{3.17}$$

Actually, they showed that the result (3.17) holds for all $n \in S_k^*$ with $\omega(n) = r$.

Pradad and the author [15] obtained an estimate for $N^*(x)$ representing the number of all $n \leq x$ in S_k^* for some $k > 1$. They proved

$$N^*(x) = 0(x^{1/2} \log^2 x \cdot (\log\, \log x)^{-2}), \quad x \to \infty. \tag{3.18}$$

A similar result for the set S_k was earlier obtained by Pomerance [6].

We finally state some results on the possible forms of n in S_k for any $k > 1$. These results are due to Prasad and Rangamma [13].

(3.19) Assume that $n \in S_k$ and

$$n = p_1 p_2 \dots p_r, \quad p_1 < p_2 < \cdots < p_r$$

are primes and s is the number of $p_i \equiv -1 (\text{mod } 3)$. Note that if $p_1 = 3$, then $s = r - 1$ in view of (3.2).

(3.20) If $p_1 = 3$, then n is of the form

$$2^{14} \cdot 3^2 \cdot m + 81921$$
$$\text{or} \quad 2^{14} \cdot 3^2 \cdot m + 131073$$

according as s is even or odd.

(3.21) If $p_1 > 3$, then n is of the form

$$2^{14} \cdot 3m + 1$$
$$\text{or} \quad 2^{14} \cdot 3m + 65637$$

according as s is even or odd.

(3.22) If $p_1 > 3$ and $s < r$, then n is of the form $2^{14} \cdot 3m + 1$.

(3.23) If $p_1 > 3$ and $s = r$, then n is of the form $2^{14} \cdot 3m + 1$ if and only if s is even.

Remark 3.24. In view of result (3.17), we have the following result.

If r is an integer such that there is no n of the above form with $10^{20} < n < (r-1)^{2^{r-1}}$, then in (3.19), $\omega(n) > r$.

4. Some Known Results for the Problem (1.4)

This problem can be stated in the equivalent form

(4.1) if $\psi(n) \equiv 1 (\mathrm{mod}\ n)$ for $n > 1$, is n a prime?

More generally, we can ask

(4.2) If $\sigma^*(n) \equiv 1 (\mathrm{mod}\ n)$, is n a prime power?

Partial results for these problems, similar to those for (1.1) and (1.3) are obtained by the author in [16]. Corresponding to the sets S_k and S_k^* of the previous section, we here study the sets $T(M)$ and $T^*(M)$ defined by

$$T(M) = \{n : \psi(n) = 1 + Mn\}$$

$$\text{and} \quad T^*(M) = \{n : \sigma^*(n) = 1 + Mn\}.$$

We content ourselves stating only one result of the author [16] as a sample.

(4.3) If $n \in T(M),\ M > 1$, then

$$3|n \Longrightarrow \omega(n) \geq 185.$$

This can be further improved, using computer aided calculations to

(4.4) $n \in T(M),\ M > 1,\ 3|n \Longrightarrow \omega(n) > 2557$ and $n > (5.9)10^{10766}$

(See the author's paper [16], p. 690) for a proof.)

5. Concluding Remarks

One can easily pose Lehmer type problems that arise from other totient functions. Prasad and the author [15] already showed ([18], Theorem 1) that the Lehmer type problem for the Jordan totient $J_r(n)$, $r > 1$ is easily settled. However, the analogous problems associated with the Schemmel totient $\phi_r(n)$ for $r = 2, 3, \ldots$ have not been investigated so far. For $r = 2$, for example, the problem is the following:

(5.1) Let $p_1, \ldots, p_r$ be odd and distinct primes satisfying

$$(p_1 - 2) \ldots (p_r - 2) | ((p_1 p_2 \ldots p_r) - 2).$$

Can we conclude that $r = 1$? It may be remarked that for any integer $k \geq 2$, the relation

$$(p_1^k - 2) \ldots (p_r^k - 2) | (p_1^k \ldots p_r^k - 2)$$

for distinct odd primes $p_1 \ldots, p_r$ implies $r = 1$. This can be shown, for example, utilizing the relation $(1 - 2p^{-k}) > (1 - p^{-k})^3$ for $p > 2$. This yields

$$\prod_{i=1}^{r}(1 - 2p^{-k})^{-1} < \prod_{p}\left(1 - \frac{2}{p^k}\right)^{-1}$$

(product taken over all odd primes)

$$< \prod_{p}(1 - p^{-k})^{-3} < \zeta^3(k)(1 - 2^{-k})^3 < 2$$

for $k \geq 2$, from which we deduce that $r = 1$.

When in (5.1) we replace 2 wherever it occurs by any integer $k > 2$, we get an infinite family of unexplored, possibly unsolvable, new problems.

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The Problems Solved by Ramanujan in the Journal of the Indian Mathematical Society

Bruce C. Berndt

Between 1912 and 1914, eight solutions by Ramanujan to questions posed in the *Journal of the Indian Mathematical Society* were published. Since these solutions have not heretofore appeared elsewhere, and since some of these problems evidently motivated certain entries in his notebooks [6], in this paper, we present all eight problems and solutions and provide some commentary on them.

Much has been written about how Ramanujan worked in isolation while in India. However, an examination of Ramanujan's early papers, several early chapters in his notebooks, the problems submitted by Ramanujan to the *Journal of the Indian Mathematical Society*, and the first twenty volumes, for example, of the *Journal of the Indian Mathematical Society* show that many of Ramanujan's early interests, especially in summing infinite series and finding identities for infinite series, were interests shared by other Indian mathematicians at that time. The problems solved by Ramanujan in the *Journal of the Indian Mathematical Society* and the related entries in his notebooks are also evidence of this.

After each question, we place in parentheses the volume and page of the originally proposed problem, followed by the volume and page(s) where Ramanujan's solution appears. With minor modifications, we have adhered to the original presentations in the *Journal of the Indian Mathematical Society.* However, we have corrected numerous misprints and made minor stylistic changes. For example, in the published solutions, infinite series are usually represented by displaying their first few terms. For brevity and clarity, we utilize summation notation. We also employ the familiar notation

$$(a)_0 := 1, \quad (a)_n := a(a+1)(a+2)\cdots(a+n-1),$$

where n is a positive integer.

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Question 336. (3 (1911), 249; 4 (1912), 35–36) (K.J. Sanjana, M.A.) *Show that*

(i)

$$\int_0^{\pi/2} (\log \sin \theta)^r \, d\theta = (-1)^r r! \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n!(2n+1)^{r+1}};$$

(ii)

$$\int_0^{\pi/2} (\log \sin \theta)^r \sec \theta \, d\theta = (-1)^r r! \sum_{n=0}^{\infty} \frac{1}{(2n+1)^{r+1}};$$

(iii)

$$\int_0^{\pi/2} (\log \sin \theta)^r \frac{1 + \tan \frac{1}{2}\theta}{1 - \tan \frac{1}{2}\theta} d\theta = (-1)^r r! \sum_{n=0}^{\infty} \frac{1}{n^{r+1}}.$$

Also evaluate the integrals when r is odd. [$(-1)^r$ *on the right side will disappear on putting log csc θ on the left side.*]

Solution by S. Ramanujan and R. Srinivasan, M.A.

(i)

$$\int_0^{\pi/2} (\log \sin \theta)^r \, d\theta = \int_0^1 \frac{(\log x)^r}{\sqrt{1 - x^2}} dx.$$

Changing x to e^{-y} the integral becomes

$$\begin{aligned} \int_0^{\infty} \frac{(-y)^r}{\sqrt{e^{2y} - 1}} dy &= (-1)^r \int_0^{\infty} y^r \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n e^{-(2n+1)y}}{n!} dy \\ &= (-1)^r r! \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n!(2n+1)^{r+1}}, \end{aligned}$$

since

$$\int_0^{\infty} y^r e^{-ky} \, dy = \frac{r!}{k^{r+1}}.$$

(ii) Similarly,

$$\begin{aligned} \int_0^{\pi/2} (\log \sin \theta)^r \sec\theta \, d\theta &= \int_0^{\infty} \frac{(-y)^r}{e^y - e^{-y}} dy \\ &= (-1)^r r! \sum_{n=0}^{\infty} \frac{1}{(2n+1)^{r+1}} \\ &= (-1)^r (2^{r+1} - 1) \pi^{r+1} \frac{B_{r+1}}{2(r+1)}. \end{aligned}$$

(iii)

$$\begin{aligned}\int_0^{\pi/2} (\log\sin\theta)^r \frac{1+\tan\frac{1}{2}\theta}{1-\tan\frac{1}{2}\theta} d\theta &= \int_0^{\pi/2} (\log\sin\theta)^r \frac{\cos\theta}{1-\sin\theta} d\theta \\ &= \int_0^{\infty} \frac{(-y)^r}{e^y-1} dy \\ &= (-1)^r r! \sum_{n=0}^{\infty} \frac{1}{n^{r+1}} \\ &= (-1)^r (2\pi)^{r+1} \frac{B_{r+1}}{2(r+1)}.\end{aligned}$$

Commentary. It is tacitly assumed that r is a nonnegative integer. In the last steps of parts (ii) and (iii), either Ramanujan assumes that r is a positive odd integer or, without comment, he is using his definition of extended Bernoulli numbers, the theory of which can be found in his notebooks [6], [1, pp. 125–127].

This first problem solved by Ramanujan in the *Journal of the Indian Mathematical Society* motivated his second published paper, *On Question* 330 *of Professor Sanjana* [4], [5, pp. 15–17]. Further related material can be found in Chapters 9 and 10 of Ramanujan's second notebook [1, pp. 237–242], [2, pp. 30–32].

We have corrected eight misprints in the published solution.

Question 324. (3 (1911), 209; 4 (1912), 111–112) (R. Srinivasan, M.A.) *Find the value of*

$$\sum_{m=1}^{\infty} \frac{(-1)^m \cos mx}{(m+1)(m+2)}.$$

Solution by V.B. Naik, S. Ramanujan and G. Ramachandran.

Writing

$$P = \sum_{m=1}^{\infty} \frac{(-1)^m \cos mx}{(m+1)(m+2)}$$

and

$$Q = \sum_{m=1}^{\infty} \frac{(-1)^m \sin mx}{(m+1)(m+2)};$$

we have

$$P + iQ = \sum_{m=1}^{\infty} \frac{(-1)^m e^{imx}}{(m+1)(m+2)}.$$

Now, the series

$$\begin{aligned}\sum_{m=1}^{\infty}\frac{(-1)^m z^m}{(m+1)(m+2)} &= \sum_{m=1}^{\infty}(-1)^m\left(\frac{1}{m+1}-\frac{1}{m+2}\right)z^m\\ &= \sum_{m=1}^{\infty}(-z)^m\frac{1}{m+1}-\sum_{m=1}^{\infty}(-z)^m\frac{1}{m+2}\\ &= \frac{\log(1+z)}{z}-1+\frac{\log(1+z)}{z^2}-\frac{1}{z}+\frac{1}{2}\\ &= \log(1+z)\left\{\frac{1}{z}+\frac{1}{z^2}\right\}-\frac{1}{z}-\frac{1}{2}.\end{aligned}$$

Putting $z = e^{ix}$, we have

$$\begin{aligned}P+iQ &= \log(1+e^{ix})\{e^{-ix}+e^{-2ix}\}-e^{-ix}-\frac{1}{2}\\ &= \left[\log\left(2\cos\frac{x}{2}\right)+i\frac{x}{2}\right][\cos x+\cos 2x - i(\sin x+\sin 2x)]\\ &\quad -\cos x+i\sin x-\frac{1}{2}.\end{aligned}$$

Equating the real parts,

$$(1)\quad P=\log\left(2\cos\frac{x}{2}\right)[\cos x+\cos 2x]+\frac{x}{2}[\sin x+\sin 2x]-\cos x-\frac{1}{2}.$$

Commentary. We have corrected two misprints.

In the published solution, it must be assumed that $z \neq -1$. If $z = -1$, then

$$(2)\qquad P=\sum_{m=1}^{\infty}\frac{1}{(m+1)(m+2)}=\sum_{m=1}^{\infty}\left(\frac{1}{m+1}-\frac{1}{m+2}\right)=\frac{1}{2}.$$

Alternatively, by letting x tend to π in (1), we may also deduce (2).

Question 325. (3 (1911), 209; 4 (1912), 187) (R. Srinivasan, M.A.) *If*

$$a_n = (-1)^n\int_0^1 (-t)_n dt,$$

show that

$$\sum_{m=0}^{\infty}(-1)^m\frac{a_m}{(x)_{m+1}}=\log\left(1+\frac{1}{x}\right).$$

Solution by S. Ramanujan.

$$\begin{aligned}\sum_{m=0}^{\infty}(-1)^m\frac{a_m}{(x)_{m+1}} &= \sum_{m=0}^{\infty}\left[\int_0^1\frac{(-t)_m}{(x)_{m+1}}dt\right]\\ &= \int_0^1\frac{1}{x}\sum_{m=0}^{\infty}\frac{(-t)_m}{(x+1)_m}dt\\ &= \int_0^1\frac{dt}{t+x}=\log\left(1+\frac{1}{x}\right).\end{aligned}$$

Commentary. Ramanujan did not indicate how he summed the series under the integral sign above, but he used a special case of Gauss's theorem. Namely, if $\operatorname{Re}(c-a-b)>0$, then

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{3}$$

which, in a different form, can be found as Entry 8 in Chapter 10 in Ramanujan's second notebook [2, p. 25]. Hence, if $a=-t$, $b=1$, and $c=x+1$ in (3), we readily deduce that

$$\frac{1}{x}\sum_{m=0}^{\infty}\frac{(-t)_m}{(x+1)_m}=\frac{1}{x+t},$$

for $\operatorname{Re}x>0$, a hypothesis tacitly assumed by the proposer and Ramanujan.

Question 471. (5 (1913), 159; 5 (1913), 229–230) (K.J. Sanjana) *Prove that*

$$\sum_{n=1}^{\infty}\frac{(\frac{1}{2})_n}{n!}\frac{x^{2n}}{2n}=\log\frac{2+2\sqrt{1-x^2}}{x^2},$$

and find the sum of

$$\sum_{n=1}^{\infty}\frac{2\cdot4\cdots2n}{3\cdot5\cdots(2n+1)}\frac{x^{2n+1}}{2n+1}.$$

Solution by S. Ramanujan and N. Sankara Aiyar.

The question should be

$$\sum_{n=1}^{\infty}\frac{(\frac{1}{2})_n}{n!}\frac{x^{2n}}{2n}=\log\frac{2}{1+\sqrt{1-x^2}}.$$

Denote the left side of the proposed identity by S. Letting $t = \sin 2\phi$ and $\theta = \frac{1}{2}\sin^{-1}x$ below, we find that

$$\begin{aligned} S &= \int_0^x \left(\frac{1}{\sqrt{1-t^2}} - 1\right)\frac{dt}{t} \\ &= 2\int_0^\theta \tan\phi\, d\phi = -2\log\cos\theta \\ &= \log\frac{2}{1+\cos 2\theta} = \log\frac{2}{1+\sqrt{1-x^2}}. \end{aligned}$$

Also, substituting $y = \sin\theta$ and then $z = \tan\frac{1}{2}\theta$, we find that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{2\cdot 4\cdots 2n}{3\cdot 5\cdots(2n+1)}\frac{x^{2n+1}}{2n+1} &= \int_0^x \frac{\sin^{-1}y}{y\sqrt{1-y^2}}dy \\ &= \int_0^{\sin^{-1}x} \frac{\theta}{\sin\theta}d\theta \\ &= 2\int_0^{x_1} \frac{1}{z}\tan^{-1}z\,dz, \end{aligned} \tag{4}$$

where

$$x_1 = \frac{x}{1+\sqrt{1-x^2}}.$$

Commentary. We have slightly rewritten the solution by Ramanujan and N. Sankara Aiyar, as well as their companion result.

For the companion result, we provide details for the initial step. Recall the expansion [3, p. 61]

$$(\sin^{-1}x)^2 = \sum_{n=0}^{\infty} \frac{2^{2n}(n!)^2 x^{2n+2}}{(2n+1)!\,(n+1)}, \qquad |x| \le 1. \tag{5}$$

Now, integrating by parts and using (5), we find that

$$\begin{aligned} \int_0^x \frac{\sin^{-1}y}{y\sqrt{1-y^2}}dy &= \int_0^x \frac{1}{y}\sin^{-1}y\frac{d}{dy}\sin^{-1}y\,dy \\ &= \frac{(\sin^{-1}x)^2}{2x} + \int_0^x \frac{(\sin^{-1}y)^2}{2y^2}dy \\ &= \sum_{n=0}^{\infty} \frac{2^{2n-1}(n!)^2x^{2n+1}}{(2n+1)!\,(n+1)} + \sum_{n=0}^{\infty} \frac{2^{2n-1}(n!)^2x^{2n+1}}{(2n+1)!\,(n+1)(2n+1)} \end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{2^{2n}(n!)^2 x^{2n+1}}{(2n+1)!\,(2n+1)}$$

$$= x + \sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdots 2n}{3 \cdot 5 \cdots (2n+1)} \frac{x^{2n+1}}{2n+1}.$$

The upper limit on the far right side of (4) in the published solution is given as $\frac{1}{2}\theta$. Perhaps tan $\frac{1}{2}\theta$ was intended. We have also corrected a couple further misprints.

Question 472. (5 (1913), 159; 5 (1913), 230–231) (Selected) *Evaluate*

$$\int_0^{\infty} \frac{\cos x}{(1+x^2)^2} dx, \quad \int_0^{\infty} \frac{\cos 2x}{(1+x^2)^2} dx.$$

Solution by R. Srinivasan, M.A. and S. Ramanujan.

We know that [Williamson, p. 151, (39)]

$$\int_0^{\infty} \frac{\cos rx}{1+x^2} dx = \frac{\pi}{2} e^{-r}.$$

Changing x into y/c, we get

$$\int_0^{\infty} \frac{\cos ry/c}{c^2+y^2} dy = \frac{\pi}{2} \frac{e^{-r}}{c},$$

or, putting a for r/c we get

$$\int_0^{\infty} \frac{\cos ay}{c^2+y^2} dy = \frac{\pi}{2} \frac{e^{-ac}}{c}.$$

Differentiating both sides with regard to c, we get

$$\int_0^{\infty} \frac{\cos ay}{(c^2+y^2)^2} dy = \frac{\pi}{4} e^{-ac} \frac{1+ac}{c^3}.$$

Putting $c = a = 1$, we get

$$\int_0^{\infty} \frac{\cos y}{(1+y^2)^2} dy = \frac{\pi}{2} e^{-1}.$$

Putting $c = 1, a = 2$, we get

$$\int_0^{\infty} \frac{\cos 2y}{(1+y^2)^2} dy = 3\frac{\pi}{4} e^{-2}.$$

Commentary. The values for these two integrals can be found in Gradshteyn and Ryzhik's *Table* [3, pp. 445, 449].

Two further solutions were also published. The third solver indicates how to evaluate

$$\int_0^\infty \frac{\cos(px)}{(1+x^2)^n}dx, \quad n = 1, 2, \ldots, \tag{6}$$

by contour integration. In fact, by further differentiations with respect to c in the solution by Ramanujan and Srinivasan, we can evaluate (6) for any $n \geq 3$. The general formula

$$\int_0^\infty \frac{\cos(ax)}{(c^2+x^2)^n}dx = \frac{\pi e^{-ac}}{(2c)^{2n-1}(n-1)!}\sum_{k=0}^{n-1}\frac{(2n-k-2)!(2ac)^k}{k!(n-k-1)!}$$

can be found in [3, p. 451].

Still another solution is given by N. Sankara Aiyar in volume 6 on pages 25–26.

Question 493. (5 (1913), 200; 5 (1913), 238) (S.P. Singaravelu Modeliar) *Sum the following series:*

(1)

$$\sum_{n=0}^\infty \frac{(\frac{1}{2})_n}{n!\,(2n+1)^2},$$

(2)

$$\sum_{n=0}^\infty \frac{(\frac{1}{2})_n}{n!\,(2n+1)^3},$$

(3)

$$\sum_{n=1}^\infty \frac{(\frac{1}{2})_n s_n}{n!\,(2n-1)(2n+1)},$$

where

$$s_n = \sum_{k=1}^n \frac{1}{(2k-1)^2}.$$

Solution by S. Ramanujan and R. Srinivasan.

Results (1) and (2) have been found to be

$$\frac{\pi}{2}\log 2 \quad \text{and} \quad \frac{\pi^3}{48} + \frac{\pi}{4}(\log 2)^2.$$

(*Vide*: Vol. IV, pages 33 and 60.)

(3) We know

$$\frac{(\sin^{-1}x)^3}{6}=\sum_{n=1}^{\infty}\frac{(\frac{1}{2})_n}{n!}\frac{x^{2n+1}}{2n+1}s_n.$$

Hence

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(\frac{1}{2})_n}{n!}\frac{x^{2n-1}}{(2n-1)(2n+1)}s_n &= \int_0^x \frac{(\sin^{-1}t)^3}{6t^3}dt\\ &= -\frac{(\sin^{-1}x)^3}{12x^2}+\int_0^x\frac{(\sin^{-1}t)^2}{4t^2\sqrt{1-t^2}}dt\\ &= -\frac{(\sin^{-1}x)^3}{12x^2}-\frac{\sqrt{1-x^2}}{4x}(\sin^{-1}x)^2\\ &\quad+\int_0^x\frac{\sin^{-1}\mathrm{t}}{2t}dt.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(\frac{1}{2})_n s_n}{n!\,(2n-1)(2n+1)} &= \int_0^1\frac{(\sin^{-1}x)^3}{6x^3}dx\\ &= -\frac{\pi^3}{96}+\frac{1}{2}\int_0^1\frac{\sin^{-1}x}{x}dx\\ &= -\frac{\pi^3}{96}+\frac{\pi\log 2}{4},\end{aligned}$$

since

$$\int_0^1\frac{\sin^{-1}x}{x}dx=\frac{\pi}{2}\log 2,$$

by (1).

Commentary. It should be noted that page 60 in volume IV is a page in Ramanujan's second published paper [4], [5, pp. 15–17]. Indeed, both (1) and (2) are special cases of a more general theorem proved by Ramanujan therein.

The Taylor series quoted at the beginning of the solution of (3) is given by Ramanujan in Section 15 of Chapter 9 in his second notebook [6], [1, p. 263]. See also Gradshteyn and Ryzhik's *Table* [3, p. 61].

The claim implied in the remark at the end of the solution, namely,

$$\int_0^1\frac{\sin^{-1}x}{x}dx=\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n}{n!\,(2n+1)^2},$$

is easily obtained by integrating the Maclaurin series for $\frac{\sin^{-1}\mathrm{x}}{x}$ termwise.

We have corrected four misprints in the published solution.

Question 490. (5 (1913), 200; 6 (1914), 34–35) (R. Srinivasan, M.A.) *Show that*

$$\sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{(2n+1)\Gamma(n+\frac{3}{2})} = \sum_{n=0}^{\infty} \left\{ \frac{4}{\sqrt{\pi}} \frac{(-1)^n}{(2n+1)^2} \right\}.$$

Solution by S. Ramanujan.

The question should be as printed above.

Putting first $x = \sin 2\theta$ and then setting $y = \tan\theta$, we find that the left side above becomes

$$\frac{2}{\sqrt{\pi}} \left\{ 1 + \sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdots 2n}{1 \cdot 3 \cdots (2n-1)(2n+1)^2} \right\}$$

$$= \frac{2}{\sqrt{\pi}} \int_0^1 \frac{\sin^{-1} x}{x\sqrt{1-x^2}} dx$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\pi/4} \frac{4\theta \cos 2\theta}{\sin 2\theta \cos 2\theta} d\theta$$

$$= \frac{4}{\sqrt{\pi}} \int_0^1 \frac{1}{y} \tan^{-1} y \, dy$$

$$= \sum_{n=0}^{\infty} \left\{ \frac{4}{\sqrt{\pi}} \frac{(-1)^n}{(2n+1)^2} \right\}.$$

Commentary. There are some misprints in the original formulation of the problem in volume 5.

A second solution was also published.

The first step in the proof is identical to that in the companion result to Question 471.

Recall that Catalan's constant G is defined by

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}.$$

Thus, after some elementary manipulation, Question 490 may be rewritten in the form

$$\sum_{n=0}^{\infty} \frac{2^{2n}(n!)^2}{(2n)!\,(2n+1)^2} = 2G. \tag{7}$$

In the formulation (7), Question 490 is identical to Example (i) in Section 32 of Chapter 9 in Ramanujan's second notebook [6], [1, p. 289]. As indicated in [1],

(7) is the special case $x = 1$ of the more general result

$$\sum_{n=0}^{\infty} \frac{2^{2n}(n!)^2}{(2n)!\,(2n+1)^2}\left(\frac{4x}{(1+x)^2}\right)^n = (1+x)\sum_{n=0}^{\infty}\frac{(-x)^n}{(2n+1)^2},$$

given as Corollary (i) in the same section [1, p. 288].

Question 492. (5 (1913), 200; 6 (1914), 36) (K.V. Anantnarayan Sastri, B.A.) *Expand* $\theta \tan \frac{\theta}{2}$ *in powers of* $\sin\theta$.

Solution by T.P. Trivedi, M.A., LL.B., and S. Ramanujan.

$$\begin{aligned}\theta\tan\frac{\theta}{2} &= \frac{\theta\sin\theta}{1+\cos\theta} = \frac{x\sin^{-1}x}{1+\sqrt{1-x^2}}\\ &= \frac{\sin^{-1}x(1-\sqrt{1-x^2})}{x},\end{aligned}$$

if $x = \sin\theta$. Now

$$\frac{\sin^{-1}x}{x} = \sum_{n=0}^{\infty}\frac{(2n)!x^{2n}}{2^{2n}(n!)^2(2n+1)}.$$

Again

$$\frac{\sin^{-1}x}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty}\frac{2^{2n}(n!)^2x^{2n+1}}{(2n+1)!}$$

Therefore,

$$\begin{aligned}\frac{(\sin^{-1}x)\sqrt{1-x^2}}{x} &= \sum_{n=0}^{\infty}\frac{2^{2n}(n!)^2x^{2n}}{(2n+1)!}(1-x^2)\\ &= \sum_{n=0}^{\infty}\frac{2^{2n}(n!)^2x^{2n}}{(2n+1)!} - \sum_{n=1}^{\infty}\frac{2^{2n-2}((n-1)!)^2x^{2n}}{(2n-1)!}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{(\sin^{-1}x)(1-\sqrt{1-x^2})}{x} &= \sum_{n=0}^{\infty}\frac{(2n)!x^{2n}}{2^{2n}(n!)^2(2n+1)}\\ &\quad -\sum_{n=0}^{\infty}\frac{2^{2n}(n!)^2x^{2n}}{(2n+1)!} + \sum_{n=1}^{\infty}\frac{2^{2n-2}((n-1)!)^2x^{2n}}{(2n-1)!}\\ &= \sum_{n=1}^{\infty}\left(\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)(2n+1)}\right.\\ &\quad \left.+\frac{2\cdot 4\cdots(2n-2)}{1\cdot 3\cdots(2n-1)(2n+1)}\right)x^{2n}.\end{aligned}$$

Replacing x by $\sin\theta$, the result is obtained.

Commentary. We have slightly reformulated the solution by Trivedi and Ramanujan. In particular, we have added one step to the last equality in the published solution, and, for ease in calculation, we have written the summands using factorials, instead of products of integers, as in the final displayed sum.

We have corrected one misprint in the published solution.

A second solution was also published.

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On the gcd and lcm of Matrices Over Dedekind Domains

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1. Introduction

In an interesting paper published in 1986, Thompson [8] gave a brief and elegant account of classical theory (traceable to the work of Cahen 1914 and Châtlet 1924, conveniently through the book of McDuffee [3]). Earlier Hua [2] had dealt with the subject in his book. Thompson also answered some questions of Morris Newman regarding relations between matrices A and B and their gcd's and lcm's in case the underlying ring R is a *principal ideal domain* (PID). Some of these are repeated here, albeit with different proofs (but for the last section, we too work over PID's).

The first thing we observe is that for A and B such that the augmented matrix

$$D = (A\ B) \tag{1}$$

may be defined, D is a gcld (l for left). Incidently, the proof works for any ring R with identity. Any two gcld's being *right equivalent* (each is a left divisor of the other), the classical theory sought a 'nice' representative of the equivalence class. We show that D defined by (1) has many nice properties.

Siegel [7] during the course his investigations into the theory of quadratic forms over the ring of integers of an algebraic number field developed a method of dealing with rectangular and other singular matrices. This was later extended to *Dedekind domains* (DD's) by Bhandari [1], and applies in particular to PID's. Using these ideas we show that if the *ranks* $r(A)$ etc. satisfy the condition

$$r(A) = r(B) = r(D), \tag{2}$$

the expression (1) leads to

(i) alternative proofs of Theorems 1 and 2 of the paper of Thompson, under reference (Theorem 2 requires minor correction, though),

(ii) proofs of existence of lcrm (r for right) via appropriate extensions of gcd-lcm duality of elementary number theory,

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(iii) alternative proofs of Theorems 3 and 4 of Thompson (the latter after a correction matching the one in Theorem 2), using duality and Theorems 1 and 2,

(iv) extensions of different formulations of elementary duality, including one in case

$$r(A) = r(D), \tag{3}$$

and an interesting one in case

$$AB = BA, \tag{4}$$

yielding a formulation that is perhaps the simplest even in case of elementary duality,

(v) deeper results on relations between various Smith invariants, leading to stronger duality under certain conditions.

In Section 2, we give, without proof, some basic results of relevant matrix theory. These include Siegel's definition of generalized inverses. We shall refer to these as Siegel inverses [4].

2. Statement of Some Results in Matrix Theory

Let R denote a Dedekind domain, and let S denote its field of quotients. We shall refer to elements of R as *integral elements*. Let $m(S)$ and $m(R)$ denote the *sets of all matrices* over S and R respectively. Elements of $m(R)$ will be referred to as *integral matrices.*

Corresponding to any $A \in m(S)$, there exits G such that ([7], [1]),

$$r(G) = r(A) = r, \text{ say; and } AG = A \tag{5}$$

G will be called *right unit* of A (*integral right unit in case G is integral*). *A left unit* similarly defined will usually be denoted as G^*.

The units are idempotents.

Given A and a prescribed G^*, there exists X such that

$$AX = G^*. \tag{6}$$

Such an X will be called a G^* *inverse* of A. The additional condition

$$GX = X \tag{7}$$

fixes X uniquely. This will be called a *generalized G^*, G inverse* of A, reference to the units being dropped at times. If G^* and G are integral, X will be called a

Siegel inverse. X, as defined above is actually a *right inverse*, but it is easily shown that it is also left inverse; moreover, $XG^* = X$, and the G, G^* inverse of X is A.

We shall denote this uniqely determined inverse as A^{-1}, noting that it is the usual inverse in case A is non-singular.

Let G denote any idempotent matrix. A matrix Y is said to be G *reduced* (on the left) in case $GY = Y$. Any GY is G reduced. The equation $AY = B$ implies $Y = A^{-1}B$ if and only if Y is G reduced, G beging a right unit of A (cancellation).

For each $i = 1, \dots, r(= (r(A)))$, we define $\Delta_i(A)$ to be the R-module generated by all i-rowed subdeterminants of A. This is a fractional ideal called *ith discriminantal divisor*. In case R is a PID, it has a single generator $\delta_i(A)$. We choose it to lie in a multiplicatively closed set of representatives of equivalence classes of associates.

(8) $\delta_i(A) = a_1 \dots a_i, a_1, \dots, a_r$ being Smith invariants of A.

We write

$$\delta_r(A) = \delta(A)$$

and call it the *discriminant* of A. If A is non-singular, $\delta(A)$ ia an associate of the determinant of A. The product rule

$$\delta(AB) = \delta(A)\delta(B)$$

holds in case A and B possess a *common integral unit* (right unit of A and left unit of B, simultaneously).

The equation $GG = G$ implies that $\delta(G) = 1$, and hence that $\delta(A^{-1}) = (\delta(A))^{-1}$. For an integral matrix P to have an integral Siegel inverse, the condition $\delta(P) = 1$ is both necessary and sufficient. Such a matrix is called *primitive*.

If X is integral, it is obvious that $\delta_i(A)|\delta_i(AX)$. In particular, $\delta_i(A) = \delta_i(B)$ in case A re B (right equivalent to), also in the more special situation of two-sided equivalence ($A \sim B$).

We write E for identity and O for zero matrices of unspecified sizes. Whenever $A + B$ or AB or augmented matrix $(A\ B)$ or block decomposition etc. occur, row/column compatibility will be assumed.

3. Greatest Common Divisors

Theorem 3.1. *Let R denote a* PID *and let S denote its field of quotients. Let $A, B \in m(S)$ and suppose that $A \neq 0$. Then A, B possess gcld if and only if they have the same number of rows.*

Proof. The necessity is obvious. Sufficiency is deduced from the simple equation

$$(A\ B)E = (A\ B), \tag{9}$$

since the block decomposition $E = (H\ K)$ implies $DH = A$ and $DK = B$, whereas the block decomposition $E = \begin{pmatrix} M \\ N \end{pmatrix}$ gives

$$AM + BN = D. \tag{10}$$

The proof follows.

Remarks 3.2. (a) In case A and B have the same number of columns, $\begin{pmatrix} A \\ B \end{pmatrix}$ would be a gcrd.

(b) Equation (10) is a simple proof even in the case of elementary number theory of the fact that $AX + BY = T$ has an integral solution if and only if gcld (A, B) is a left divisor of T.

(c) The augmented matrix $(A\ B_1\ B_2 \dots B_m) = ((A\ B_1)\ B_2 \dots B_m)$ is a gcld of $A, B_1, \dots, B_m$.

(d) For any non-zero a, gcld $(aA, aB) = a.gcld(A, B)$, so that we might, *whenever useful, assume A, B integral.*

(e) If U, V, W are appropriately reduced and primitive, then $UAV = A_1$, $UBW = B_1$ and $U(A\ B)\begin{pmatrix} V & 0 \\ 0 & W \end{pmatrix} = D_1$ have the same Smith invariants as A, B and D respectively.

This is interesting, because in case R is a PID, U, V, W may be so chosen that UAV is in *Smith Normal Form (SNF)* $= \operatorname{diag}[a_1, \dots, a_r], a_1|a_2, \dots, a_{r-1}|a_r$, $a_r \neq 0$; while simultaneously UBW is lower triangular *Hermite Normal Form (HNF)* $= (b_{ij})$.

We are now ready for an alternative proof of Thompson's Theorem 1.

Theorem 3.3. *If* $a_1, \dots, a_r; b_1, \dots, b_s, r \geq s$; *and* $d_1, \dots, d_r$ *are respective Smith invariants of A, B and D, then (with* $b_{s+1} = 0 = \cdots = b_r$*),*

(i) $(a_i, b_j)|d_{i+j-1}$, *and*

(ii) $d_i|(a_i, b_i)$.

Proof. In view of Remark 3.2 (c, d, e), it suffices to prove the theorem in the case A, B are integral, and

$$A = \operatorname{diag}[a_1, \dots, a_r], a_1|a_2, \dots, a_{r-1}|a_r, a_r \neq 0,\ B = \begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix}.$$

Let $\delta_1(B) = \delta(B) = b$, then a simple observation shows that $(a_1, b) = d_1$.

Since $b_j = 0$ for $j > 1$, we need only prove that

$$(a_i, b) | d_i \text{ and } d_i | a_i \tag{11}$$

Now $\delta_i(D) = (a_1 \ldots a_i, a_1 \ldots a_{i-1}(b_i, \ldots, b_r), a_2 \ldots a_i(b_1, b_{i+1}, \ldots, b_r), \ldots)$. Thus

$$\delta_i(D) = (\delta_i(A), a_1^{-1}\delta_i(A).b), \text{ and}$$

$$(\delta_i(A), \delta_{i-1}(A).b) | \delta_i(D).$$

In order to prove (11), it suffices to prove that

$$(\delta_{i-1}(A), \delta_{i-2}(A).b)(a_i, b) | ((\delta_i(A), \delta_{i-1}(A)b), \text{ and}$$

$$(\delta_i(A), a_1^{-1}\delta_i(A).b) | a_i(\delta_{i-1}(A), a_1^{-1}b\delta_{i-1}(A))$$

and both are obvious. □

A technically more complicated but direct proof can also be given.

4. Least Common Multiples, Duality, Applications

We start off by recalling the following different formulations of gcd-lcm duality of elementary number theory (referred to hereafter as *elementary duality*), and then obtain, at times under restrictions, corresponding matrix theoretic dualities.

Remark 4.1. Let (a, b) and $[a, b]$ denote respectively the gcd and lcm of $a, b \in S$. Then

(i) $(a, b)[a, d] = ab$,
(ii) $[a, b] = (a^{-1}, b^{-1})^{-1}$,
(iii) $[a, b] = ax = \text{by}$ if and only if $(x, y) = 1$,
(iv) $[a, b] = ab$ in case $(a, b) = 1$,
(v) (we don't find this one in books on elementary number theory)

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \sim \begin{pmatrix} (a, b) & 0 \\ 0 & [a, b] \end{pmatrix}.$$

Suppose that A and B are invertible, then over a PID, every cld of A and B is right equivalent to an invertible matrix C. It is obvious that C^{-1} is an lcrm of A^{-1} and B^{-1} if and only C is a gcld of A and B. Remark 3.2(e) shows that over a PID, a more general situation can be reduced to this. However, partly because of its elegance and partly because of natural extension to Dedekind domains (where such reduction to non-singular matrices is not always possible), we give a formulation using Siegel inverse that omits this apparent simplification.

Theorem 4.2. *Let A, B have a simultaneous integral left unit I, then any Siegel inverse M of $N = \begin{pmatrix} A^{-1} \\ B^{-1} \end{pmatrix}$, where the inverses are all with respect to I, is an lcrm of A and B.*

Proof. I is obviously a simultaneous right unit of A^{-1} and B^{-1}, so that $\operatorname{gcrd}(A^{-1}, B^{-1})$ exists. If $M_1 = AX = BY$ is any crm, then $M^{-1}M_1 = \begin{pmatrix} X \\ Y \end{pmatrix}$, with I-reduced $\begin{pmatrix} X \\ Y \end{pmatrix}$.

That proves the theorem. □

We have thus extended 4.1(ii).

Corollary 4.3. (a) *Theorem 1 of Thompson (Th. 3.3) is equivalent to Th. 3 of Thompson: Let $m_1, \dots, m_r$ denote Smith invariants of M. Then with the notation of Th. 3.3, $[a_i, b_i] | m_i$ and $m_{i+j-r} | [a_i, b_j]$.*

(b) *Theorems 2 and 4 of Thompson corrected as follows are equivalent:*

Corrected Th. 2: Given $d_1, \dots, d_r$ satisfying $d_1 | d_2, \dots, d_{r-1} | d_r$, fix D with these invariants. Select any $a_i, b_i, i = 1, \dots, r$, satisfying $a_{i-1} | a_i, b_{i-1} | b_i$, $d_i | (a_i, b_i) | d_{2i-1}$. Then there exist A, B with invariants $a_1, \dots, a_r; b_1, \dots, b_r$ such that D is their gcld.

Corrected Th. 4: Given $m_1, \dots, m_r, m_{i-1} | m_i$, fix M with these invariants. Select any $a_i, b_i, i = 1, \dots, r$ satisfying $a_{i-1} | a_i, b_{i-1} | b_i, m_{2i-r} | [a_i, b_i] |$ m_i respectively. Then there exist A, B with invariants $a_1, \dots, a_r; b_1, \dots, b_r$ such that M is their lcrm.

Proof. We notice that $a_r^{-1}, \dots, a_1^{-1}$; $b_r^{-1}, \dots, b_1^{-1}$ and $m_r^{-1}, \dots, m_1^{-1}$ are the Smith invariants of A^{-1}, B^{-1} amd M^{-1} respectively. By Th. 3.3,

(i) $m_i^{-1} | (a_i^{-1}, b_i^{-1})$, which is the same as $[a_i, b_i] | m_i$,

(ii) $(a_{r-i-1}^{-1}, b_{r-i-1}^{-1}) | m_{r-i-j-1}^{-1}$, which is equivalent to $m_{i+j-r} | (a_i, b_j)$.

□

Theorem 4.4. *Let A, B possess a simultaneous left and a simultaneous right unit as well as a common integral unit, then with M_1 denoting lclm (A, B)*

$$\delta(A)\delta(B) = \delta(D)\delta(M_1). \tag{12}$$

Proof. We note that

$$AB(A^{-1}\, B^{-1}) = (ABA^{-1}\, A). \tag{13}$$

In view of 3.2(e), for evaluation of discriminants, we may assume that $A = \operatorname{diag}[a_1, \dots, a_r]$ is in *SNF* and B is lower triangular. Writing $ABA^{-1} = T$, we notice that T is also lower triangular and that the r-rowed subdeterminants of $(T\ A)$ and $(B\ A)$ are the same. The result follows on consideration of discriminants of two sides of (13).

This generalizes 4.1(i), although for special A, B.

The special case $AB = BA$ of the above theorem is interesting. For, then

$$AB(A^{-1}B^{-1}) = (BA), \tag{14}$$

which demonstrates duality in perhaps the simplest possible way. The rank one subcase is 4.1(i).

Suppose A and B possess a simultaneous right unit as well as a simultaneous left unit, then they have a gcrd and a gcld. Because matrix multiplication is not commutative, there is little reason to expect these to be equivalent. However, since a matrix and its transpose are equivalent (over a PID), in case it is possible to find reduced and primitive P and Q such that PAQ and PBQ are symmetric or skew symmetric, these are equivalent. Another result in the positive direction is stated in the form of the next theorem.

Theorem 4.5. *Let* $r(A) = r(B) = r((A\ B)) = r\left(\begin{pmatrix} A \\ B \end{pmatrix}\right)$, *then with the notation*

$$C = \begin{pmatrix} A & O \\ O & B \end{pmatrix}, K = \begin{pmatrix} D & O \\ O & M \end{pmatrix}, L = \begin{pmatrix} D_1 & O \\ O & M_1 \end{pmatrix}, \tag{15}$$

$\delta_i(C)|(\delta_i(K), \delta_i(L))$.

Proof. $C \sim \begin{pmatrix} A & B & O \\ O & B & M \end{pmatrix} = N$ say; since $C \sim \begin{pmatrix} A & B \\ O & B \end{pmatrix}$, obviously and if $M = AX = BY$, $\begin{pmatrix} A & B \\ O & B \end{pmatrix}\begin{pmatrix} E & O & X \\ O & E & Y \end{pmatrix} = \begin{pmatrix} A & B & O \\ O & B & M \end{pmatrix}$ and $N\begin{pmatrix} E & O \\ O & E \\ O & O \end{pmatrix}$ $= \begin{pmatrix} A & B \\ O & B \end{pmatrix}$. Now $\begin{pmatrix} A & O \\ O & M \end{pmatrix}$ as well as $\begin{pmatrix} B & O \\ B & M \end{pmatrix} \sim \begin{pmatrix} B & O \\ O & M \end{pmatrix}$ are submatrices of N, hence $\delta_i(N)(= \delta_i(C))|\delta_i(K)$. A similar argument with L gives the result. □

4.1(v) does not extend, in general, to matrices. The next theorem gives a set of sufficient conditions under which such stronger duality holds.

Theorem 4.6. *With notation explained in* (15), $C \sim K$ *in case any one of the following conditions is satisfied:*

- (i) A *ld* B,
- (ii) $(a_i, b_i) = d_i$, *where* a_i, b_i *and* d_i *denote Smith invariants of* A, B *and* D *respectively,*
- (iii) A *is a scalar matrix.*

Proof. (i) Obvious since gcld $= A$ and lcrm $= B$.

(ii) Let $C_1 \sim \begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}$, and $K_1 \sim \begin{pmatrix} d_1 & 0 \\ 0 & m_1 \end{pmatrix}$. Defining C_2, K_2 by $C \sim \begin{pmatrix} C_1 & O \\ O & C_2 \end{pmatrix}$, $K \sim \begin{pmatrix} K_1 & O \\ O & K_2 \end{pmatrix}$, we see that $C \sim K$ if and only if $C_2 \sim K_2$. We use induction to finish off the argument.

(iii) Since the Smith invariants over R are products of all local smith invariants, it suffices to prove the result in case the underlying ring is R_p, the localization at the prime p. Let $A = p^a E$ and $B \sim \operatorname{diag}[p^{b_1}, \dots, p^{b_r}]$. Let s be defined by $b_s \leq a < b_{s+1}$ in case $a \leq b_r$ and $s = r$ in case $a > b_r$. Then $\operatorname{gcld}(A, B) \sim \operatorname{diag}[p^{b_1}, \dots, p^{b_s}]$ and lcrm $(A, B) \sim \operatorname{diag}[p^a, \dots, p^a, p^{b_{s+1}}, \dots, p^{b_r}]$ and the proof is complete. □

In the next theorem, we obtain lcrm(A, B) under the less restrictive condition:

$$r(A) = r((A\ B)), A \neq 0. \tag{16}$$

There is no restriction on the number of columns. Condition (16) implies that B is left reduced with respect to a left unit of A. Thus $r(B) \leq r(A)$.

Theorem 4.7. *Let A, B satisfy* (16), *then A, B possess an lcrm of rank $r(B)$.*

Before proving the theorem, we prove a simple result stated in the form of a lemma, because it has application elsewhere too.

Lemma 4.8. *Let V, W have a simultaneous left unit. Then they possess a simultaneous integral left unit.*

Proof. The existence of simultaneous left unit implies that $r(V) = r(W) = r((V\ W))$. Any integral left unit of $(V\ W)$ is then a simultaneous left unit of V and W. □

Proof of Theorem 4.7. Let s be non-zero. Clearly A, B would possess an lcrm of rank $r(B)$ if sA and sB possess one. We may assume therefore that A, B are integral.

We observe first of all that for any a such that aA^{-1} is integral, the matrix aB is a crm of rank $r(B)$, since

$$aB = B.aE = A.aA^{-1}B$$

(notice that $AA^{-1}B = B$). Let $N = AV = BW$, V, W reduced. Thus $V = A^{-1}N$ and $W = B^{-1}N$ are of the same rank and possess a simultaneous right unit (any right unit of N). Writing $U = \begin{pmatrix} V \\ W \end{pmatrix}$

$$P = UU^{-1} = \begin{pmatrix} X \\ Y \end{pmatrix} \tag{17}$$

is a primitive matrix. Define M by

$$M = NU^{-1} = AX = BY.$$

We wish to show that M is an lcrm. Let $N = AF = BG$ denote a crm. In order to show that M ld N, we show that

$$H = \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} K = PK, K \text{ integral.} \tag{18}$$

The matrix P is primitive. We need only show that H and P possess a simultaneous integral left unit, noting in view of the above lemma that the requirement of the unit being integral can be dropped.

If C is any left unit of B, then

$$I = \begin{pmatrix} 2C & -C \\ 2C & -C \end{pmatrix}$$

is a left unit of $\begin{pmatrix} N \\ N \end{pmatrix}$ as well as of $\begin{pmatrix} M \\ M \end{pmatrix}$. It follows that

$$J = \begin{pmatrix} A^{-1} & 0 \\ O & B^{-1} \end{pmatrix} I \begin{pmatrix} A & 0 \\ O & B \end{pmatrix}$$

is a left unit of P and H.

Remark 4.9. (a) We have proved that $M = AX = BY$ is an lcrm if and only if X and Y are right coprime. This extends 4.1 (iii).

(b) In case A and B as also B and A possess common integral units, and $AB = BA$, the matrix AB is an lcrm, if B is right coprime with A. This happens, for example in case $r(B) = r(A)$ and $\delta(A)$ and $\delta(B)$ are coprime.

5. The Dedekind Domain Case

R in this section will denote a DD. We follow generally the notation used in [5] but draw upon the work of Siegel, Bhandari and Narang. We will continue to use small letters for elements of S and capital letters for matrices. Since ideals, in general, may not be identified with single generators, we use small Greek letters for those (exception: δ used for discriminants).

π will stand for a prime ideal and R_π, for the localization of R at π. It is known that R_π is a PID; so that the earlier theory will apply to matrices over R_π and through local-global consideration to those over R, and S.

A direct sum of matrices of rank one will be referred to as *generalized diagonal* (we will actually deal only with 2×2 matrices of rank one). Such a matrix of rank r will thus have r pairs of rows each of rank one, also r pairs of columns each of rank one.

A matrix $T = (T_{ij}), i, j = 1, \dots, r$; where each T_{ij} is a 2×2 matrix, and $T_{ij} = 0$ for $i < j$, will be called *generalized triangular* if it satisfies the following additional conditions:

(i) $T = (T_1, T_2, \dots, T_r)$, each T_i being $r \times 2$, of rank one,
(ii) it has r dependent pairs of rows.

A generalized diagonal matrix as defined above has a generalized diagonal left (as also right) unit - there would also be other units. The generalized triangular matrices as defined above will be left as well right reduced with respect to generalized diagonal idempotents.

A matrix $A = \operatorname{diag}[A_1, \dots, A_r]$ is said to be in *GSNF* (*G* for generalized) in case $\delta(A_{i-1}) | \delta(A_i), i = 2, \dots, r$. Writing α_i for $\delta(A_i)$, we refer to $\alpha_1, \alpha_2/\alpha_1, \dots, \alpha_r/\alpha_{r-1}$ as Smith invariants. An important difference from the situation in the PID case is that two matrices with the same set of Smith invariants need not be equivalent, not even in the rank one case. Imitating 3.3(e), starting with A and B such that $(A\ B)$ makes sense, we may find U, V and W appropriately reduced and primitive and replace A, B by *UAV*, *UBW* of equivalent gcld; and in case A and B possess a simultaneous left unit, so also will *UAV* and *UBW*. Finally, U, V, W may be so chosen that *UAV* is in *GSNF* and *UBW* is lower triangular (indeed in *GHNF*), and have a simultaneous generalized left unit.

Results from the theory over PID's that go through smoothly in the new situation include 3.3, 4.1(i),..., (iv) with ideals in place of numbers; 4.2, 4.4 and 4.5 through *local-global* considerations (the discriminantal divisors of any given matrix over R_π are simply the precise powers of π that divide the corresponding discriminantal divisors over R); 4.6(i) extends with the same proof. The statement of 4.6(ii) requires modification, because although $(\delta(A_i), \delta(B_i)) = \delta(D_i)$ would imply $[\delta(A_i), \delta(B_i)] = \delta(M_i)$, where $\delta(D_i)$ and $\delta(M_i)$ denote respective invariants of $\operatorname{gcld}(A, B)$ and $\operatorname{lcrm}(A, B)$; this in turn does not imply the equivalence of C and K. A similar result holds for 4.6(iii).

A matrix and its transpose possess the same Smith invariants, but may not be equivalent, so that the unnumbered remark after Th. 4.4 does not hold.

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The Billiard Ball Motion Problem I: A Markoff Type Chain for the Octahedron in $\mathfrak{R}^3$

R.J. Hans-Gill

Dedicated to Professor R.P. Bambah on his 75th birthday

1. Introduction

Suppose that a particle moves in a rectilinear, uniform motion inside the unit cube

$$\mathbf{U} : -1/2 \le x_i \le 1/2, \quad i = 1, \ldots, n.$$

Suppose further that on striking a boundary hyperplane $x_i = \pm 1/2$ of $\mathbf{U}$, it is reflected in the usual way, like a billiard ball. The nature of the paths of such motions has been studied by various authors: König and Szücs [5], Sudan [10] (Also see Hardy and Wright [4].).

Schoenberg [6–8] considered an extremum problem related to these paths. A path is called *non-trivial* if it does not lie in a hyperplane parallel to a co-ordinate hyperplane. Let $\mathbf{C}$ be a closed convex body containing the origin O and let $\mathbf{C}^*$ denote its interior. What is the largest number $\rho(\mathbf{C})$ such that there is a non-trivial path lying in $\mathbf{U} - \rho(\mathbf{C})\mathbf{C}^*$ = the complement of $\rho(\mathbf{C})\mathbf{C}^*$ in $\mathbf{U}$? Which are the non-trivial paths lying in this complement? Such paths are called *critical* paths.

Schoenberg formulated and studied this problem for l_p-balls and boxes centered at the origin O. He obtained $\rho(\mathbf{C})$ when $\mathbf{C}$ is a box in $\mathfrak{R}^n$ or $\mathbf{C}$ is an octahedron and n is even or $\mathbf{C}$ is a circle. The result for boxes also follows from results of Wills [11]. Schoenberg [7] made a conjecture for spheres in $\mathfrak{R}^n$, which has been proved for $n = 3$ and 4 by Dumir, Hans-Gill and Wilker [2], where they also obtained Markoff type chains of successive extreme values. Some generalisations, when lines are replaced by flats, have been studied by Schoenberg [9] and Dumir, Hans-Gill and Wilker [1].

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Schoenberg also obtained lower and upper bounds for $\rho(\mathbf{C}_p)$, where

$$\mathbf{C}_p : \sum_{i=1}^{n} |x_i|^p \le 1, \quad 1 \le p < \infty$$

is an l_p-ball in $\mathfrak{R}^n$. He showed that

$$\mu_n^{(p)} \le \rho(\mathbf{C}_p) \le \frac{1}{2}\left(\frac{n}{p+1}\right)^{1/p}, \tag{1.1}$$

where

$$\mu_n^{(p)} = \begin{cases} \frac{2^{1/p}}{4m}(1^p + 3^p + \cdots + (2m-1)^p)^{1/p} & \text{for } n = 2m & (1.2) \\ \frac{2^{1/p}}{2m+1}(1^p + 2^p + \cdots + m^p)^{1/p} & \text{for } n = 2m+1 & (1.3) \end{cases}$$

He obtained the left inequality in (1.1) by considering a special path called the path of the "Lucky Shot". The Lucky Shot has $(0, \frac{1}{n}, \ldots, \frac{n-1}{n})$ as an initial point and $(1, \ldots, 1)$ as initial direction. He also observed that for even n and $p = 1$, both bounds in (1.1) are equal and hence $\rho(\mathbf{C}_1) = n/4$ for such n.

Here we consider the case $n = 3$ and $p = 1$. We show that equality holds on the left side of (1.1) i.e. $\rho(\mathbf{C}_1) = 2/3$ and the critical paths are essentially given by the Lucky Shot i.e. all critical paths are obtained from the path of the Lucky Shot by reflections in co-ordinate planes and renaming of co-ordinates.

In another paper we shall extend this result to all p-balls in $\mathfrak{R}^3$.

From some results of Dumir, Hans-Gill and Wilker [1] quoted in Lemma 4.1 of this paper, it follows that 1/2 is the largest value of ρ such that there are uncountably many paths lying in $\mathbf{U} - \rho\mathbf{C}_1^*$. We carry our investigations further to determine all paths $\mathbf{\Gamma}$ lying in $\mathbf{U}$ outside $\frac{1}{2}\mathbf{C}_1^*$. For all such paths $\mathbf{\Gamma}$, we determine the smallest expansion of $\mathbf{C}_1$ which meets $\mathbf{\Gamma}$. Thereby we obtain a Markoff type chain of paths and the corresponding sequence of successive extreme values converging to $\frac{1}{2}$. This result is analogous to that for spheres [2]. But it is in contrast with the corresponding result for 3-dimensional octahedron $\mathbf{C}_1$ for view-obstruction problem where the first extreme value is attained infinitely many times (see [3]).

In another paper we shall obtain an analogous Markoff type chain for boxes in $\mathfrak{R}^3$.

We **conjecture** that the equality holds on the left in (1.1) for all n and p. (This extends the conjecture of Schoenberg for spheres mentioned earlier). The critical paths are also conjectured to be essentially those given by the Lucky Shot, except possibly for the case $p = 1$.

2. Equivalent Formulation and Notation

It is easy to see that in case $\mathbf{C}$ is symmetric in all the co-ordinate hyperplanes, determination of $\rho(\mathbf{C})$ is equivalent to determining the supermum of $\mathbf{C}$-norm distances of $\mathbf{Z}^n$ from lines which are not parallel to any co-ordinate hyperplane (c.f. Schoenberg [8], Dumir, Hans-Gill and Wilker [1, 2]). Here we shall be working with p-balls which have these symmetries.

Let

$$\mathbf{C}_p : \|X\|_p \le 1,$$

where

$$\|X\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \quad \text{for } 1 \le p < \infty$$

is the l_p-norm.

For non-empty subsets S and S′ of $\mathfrak{R}^n$, let d_p (S, S′) be the l_p-norm distance between S and S′ i.e.

$$d_p(\mathbf{S}, \mathbf{S}') = \inf\{\|X - X'\|_p : X \in \mathbf{S}, X' \in \mathbf{S}'\}.$$

For $A \in \mathfrak{R}^n$, we write $d_p(\{A\}, \mathbf{S}) = d_p(A, \mathbf{S})$. Let $\mathbf{Z}^n$ denote the integral lattice. It is easy to see that if **L** is a line with a proper segment in common with a path $\mathbf{\Gamma}$, then

$$d_p(\mathbf{Z}^n, \mathbf{L}) = d_p(O, \mathbf{\Gamma}).$$

Therefore

$$\begin{aligned} \rho(\mathbf{C}_p) &= \sup\{d_p(O, \mathbf{\Gamma}) : \mathbf{\Gamma} \text{ non-trivial path of a billiard ball motion}\} \\ &= \sup\{d_p(\mathbf{Z}^n, \mathbf{L}) : \mathbf{L} \text{ any line through } \mathbf{U}^* \text{ not lying in a hyperplane} \\ &\qquad x_i = k\} \\ &= \sup\{d_p(\mathbf{Z}^n, \mathbf{L}) : \mathbf{L} \text{ any line not lying in a hyperplane } x_i = k\}. \end{aligned}$$

Define

$$\bar{d}_p(\mathbf{L}) = \sup\ \{d_p(\mathbf{Z}^n, \mathbf{L} + P) : P \in \mathfrak{R}^n\}.$$

Then

$$\begin{aligned} \rho(\mathbf{C}_p) &= \sup\{\bar{d}_p(\mathbf{L}) : \mathbf{L} \text{ any line through } O \text{ not lying} \\ &\qquad \text{in a coordinate hyperplane}\}. \end{aligned}$$

Notice that if the supremum in $\bar{d}_p(\mathbf{L})$ is attained at a point P_0, it is also attained at all points of $P_0 + \mathbf{Z}^n + \mathbf{L}$. It is enough to determine one representative. Any set of such representatives of all points where the supremum is attained will be denoted by $\mathbf{S}_p(\mathbf{L})$.

Any line **L** through O, which does not lie on a co-ordinate hyperplane is determined by a point A on **L**, $A = (a_1, \ldots, a_n), a_i \ne 0, 1 \le i \le n$. The vector A

also gives the direction ratios of L and we use these notions interchangeably. It is also convenient to write

$$\bar{d}_p(A) = \bar{d}_p(\mathbf{L}) \text{ and } S_p(A) = S_p(\mathbf{L}).$$

Due to the symmetries of $\mathbf{C}_p$, it follows that

$$\rho(\mathbf{C}_p) = \sup\{\bar{d}_p(A) : a_i > 0, \quad i = 1, \ldots, n\}.$$

Remark 2.1. Consider the family of sets $\{\mathbf{C}_p + X : X \in \mathbf{Z}^n\}$. Let $\phi_{\mathbf{L}}$ denote the projection parallel to a given line $\mathbf{L}$ on a given hyperplane II not parallel to a hyperplane containing $\mathbf{L}$. Then

$$\bar{d}_p(\mathbf{L}) = \inf\{\alpha > 0 : (\alpha\phi_L(\mathbf{C}_p), \phi_{\mathbf{L}}(\mathbf{Z}^n)) \text{ is a covering}\}.$$

The set $\phi_{\mathbf{L}}(\mathbf{Z}^n)$ is always closed under addition but it may not be a lattice. $\bar{d}_p(\mathbf{L})$ is the covering radius of $\phi_{\mathbf{L}}(\mathbf{C}_p)$ relative to $\phi_{\mathbf{L}}(\mathbf{Z}^n)$. A point P of a configuration $(\mathbf{S}, \mathbf{\Lambda})$ is said to be just covered if P does not lie in the interior of any set $\mathbf{S} + A$, $A \in \mathbf{\Lambda}$. When $\phi_{\mathbf{L}}(\mathbf{Z}^n)$ is a lattice, the set of just covered points in a fundamental parallelogram corresponds to an $\mathbf{S}_p(\mathbf{L})$. This formulation is useful when the covering radius is easy to determine. It has already been used in [2] for the problem of spheres in $\mathfrak{R}^3$ and $\mathfrak{R}^4$. This will be our main tool in this paper.

Remark 2.2. The formulation described in the beginning of this section connects the billiard ball motion problem to various visibility problems: view-obstruction, separability, cylindrical holes in given family of sets and covering minima of convex bodies. Since we find it convenient and more fruitful to work with this formulation, we state our results accordingly in the next section.

3. Statement of Results

Now we shall be concerned with the octahedron i.e. the case $p = 1$, in 3-dimensional space. We shall suppress the subscript 1 and write $\mathbf{C}$, $d(\mathbf{S}, \mathbf{T})$, $\bar{d}(A)$, $\mathbf{S}(A)$ for $\mathbf{C}_1$, $d_1(\mathbf{S}, \mathbf{T})$, $\bar{d}_1(A)$, $\mathbf{S}_1(A)$ respectively. We prove

Theorem 3.1.

(i) $\rho(\mathbf{C}) = 2/3$.

(ii) *The path of the Lucky shot is, essentially, the only critical path.*

This theorem can be stated in terms of the distance $\bar{d}(A)$. Because of the symmetries of $\mathbf{C}$ it is enough to consider $A = (1, w_1, w_2)$, $1 \leq w_1 \leq w_2$.

Theorem 3.1′.

(i) *For* $A_0 = (1, 1, 1)$, *we have*

$$\bar{d}(A_0) = 2/3 \textit{ and } \mathrm{S}(A_0) = \{(0, \tfrac{1}{3}, \tfrac{2}{3}), (0, \tfrac{2}{3}, \tfrac{1}{3})\}.$$

(ii) *If* $w_2 > 1$ *i.e.* $A \neq A_0$, *then* $\bar{d}(A) < 2/3$.

Now we make a detailed study to determine all A for which $\bar{d}(A) > \frac{1}{2}$ and those for which $\bar{d}(A) = \frac{1}{2}$. We find that there are countably many points A satisfying the first property and uncountably many satisfying the second.

Theorem 3.2. *For* $A = (1, w, w)$, w *any natural number,*

$$\bar{d}(A) = \frac{2w}{4w-1} > \frac{1}{2}$$

and $\mathrm{S}(A) = \{(0, \frac{w}{4w-1}, \frac{3w-1}{4w-1}), (0, \frac{3w-1}{4w-1}, \frac{w}{4w-1})\}$.

Theorem 3.3. *If* w_1 *and* w_2 *are not both rational, then*

$$\bar{d}(A) \leq \frac{1}{2}$$

and equality holds if and only if either $w_1 = w_2$ *or* $w_1 = 1$ *or* $w_2 = 1 + w_1$.

Now we consider the case when w_1, w_2 are both rational.

Theorem 3.4. *Suppose that* $w > 0$ *and* A *is either* $(1, w, w), w \in \mathbf{Q} - \mathbf{Z}$ *with* $w > 1$ *or* $(1, w, 1 + w), w \in \mathbf{Q}$. *Write* $w = a/m, a, m$ *co-prime positive integers. Then*

$$\bar{d}(A) = 1/2 \text{ and } \mathrm{S}(A) = \{(0, b + \tfrac{1}{2}, b) : 0 \leq b < \tfrac{1}{m}\}.$$

Theorem 3.5. *For* $A = (1, 1, w), w \in \mathbf{Q}, w > 1$, *we have*

$$\bar{d}(A) = \tfrac{1}{2}$$

and if we write $w = m/a$, *where a, m are relatively prime positive integers, then*

$$\mathrm{S}(A) = \begin{cases} \{(b + \frac{1}{2}, b, 0) : 0 \leq b < \frac{1}{m}\} & \text{if } m \neq 2 \\ \{(b + \frac{1}{2}, b, 0) : 0 \leq b < \frac{1}{2}\} & \\ \quad \cup\{(-b + \frac{1}{2}, b, 0) : 0 < b < \frac{1}{2}\} & \text{if } m = 2. \end{cases}$$

Theorem 3.6. *For all* $A = (1, w_1, w_2), 1 < w_1 < w_2, w_2 \neq w_1 + 1, w_1, w_2 \in \mathbf{Q}$ *we have*

$$\bar{d}(A) < \tfrac{1}{2}.$$

4. Proofs of the Results

We shall use the idea of utilizing the projection as explained in Remark 2.1. We shall project the sets $\rho\mathbf{C}+X$, $X \in \mathbf{Z}^n$, for suitable ρ, usually on the plane $x_1 = 0$. The projection function ϕ_A is given by

$$\phi_A(x_1, x_2, x_3) = (x_2 - w_1 x_1, x_3 - w_2 x_1).$$

Clearly, $\phi_A(\mathbf{Z}^3) \supset \mathbf{Z}^2$. Moreover $\phi_A(\mathbf{Z}^3)$ is an additive group. It is a lattice if and only if both w_1, w_2 are rational.

The projection of the octahedron **C** is the convex cover of the projections of the vertices $\pm(0,1,0), \pm(0,0,1), \pm(1,0,0)$. These are

$$\pm(1,0), \pm(0,1) \text{ and } \mp(w_1, w_2).$$

We denote this "hexagon" by $\mathbf{H}(w_1, w_2)$ or $\mathbf{H}(A)$. Clearly

$$\phi_A(\rho\mathbf{C}) = \rho\mathbf{H}(A) = \rho\mathbf{H}(w_1, w_2).$$

Thus each theorem here is equivalent to a result about special type of coverings $(\rho\mathbf{H}(A), \phi_A(\mathbf{Z}^3))$. Often this is easily seen by drawing a figure.

Proof of Theorem 3.1′. (i) Here $w_1 = w_2 = 1, \rho = 2/3, \phi_{A_0}(\mathbf{Z}^3) = \mathbf{Z}^2$ and $\rho\mathbf{H}(A_0)$ is a hexagon with vertices $\pm(2/3, 0), \pm(0, 2/3)$ and $\pm(2/3, 2/3)$. It is easily seen that this hexagon has $\mathbf{Z}^2$ as a covering lattice and $(1/3, 2/3)$ and $(2/3, 1/3)$ are the only just covered points in the fundamental square $0 \le x < 1, 0 \le y < 1$. Thus we can take $\mathbf{S}(A_0) = \{(0, \frac{1}{3}, \frac{2}{3}), (0, \frac{2}{3}, \frac{1}{3})\}$.

(ii) Here $w_2 > 1$. Take $\rho = 2/3$. We notice that $\rho\mathbf{H}(A)$ contains $\rho\mathbf{H}(A_0)$. Further $\rho\mathbf{H}(A)$ contains in its interior both open horizontal edges of $\rho\mathbf{H}(A_0)$. Therefore, the points $(1/3, 2/3)$ and $(2/3, 1/3)$ lie in the interiors of $\rho\mathbf{H}(A)$ and $\rho\mathbf{H}(A) + (1,1)$ respectively. Therefore $(\rho\mathbf{H}(A), Z^2)$ is a covering and there are no just covered points. Since $\mathbf{H}(A)$ is bounded there exists $\rho_0 < \rho$, ρ_0 depending on A, such that $(\rho_0\mathbf{H}(A), \mathbf{Z}^2)$ is also a covering. Since $\phi_A(\mathbf{Z}^3)$ contains $\mathbf{Z}^2$, it follows that $(\rho_0\mathbf{H}(A), \phi_A(\mathbf{Z}^3))$ is a covering and $\overline{d}(A) \le \rho_0 < \rho$. □

Remark. An alternative proof of Theorem 3.1 can be given by using the isolation results in the two dimensional case i.e. regarding the domain $|x| + |y| \le 1$.

Proof of Theorem 3.2. Since w is an integer, the projected lattice is $\mathbf{Z}^2$. Let $\rho = \frac{2w}{4w-1}$. Here the hexagon $\rho\mathbf{H}(A)$ has vertices $\pm(\rho, 0), \pm(0, \rho)$ and $\pm(\rho w, \rho w)$. We notice that the boundaries of $\rho\mathbf{H}(A)$ and $\rho\mathbf{H}(A) + (1,1)$ intersect at the points $P = (1 - \frac{1}{2}\rho, \frac{1}{2}\rho)$ and $P' = (\frac{1}{2}\rho, 1 - \frac{1}{2}\rho)$. The points P and P' also lie on the boundaries of $\rho\mathbf{H}(A) + (1,0)$ and $\rho\mathbf{H}(A) + (0,1)$ respectively. In fact, P is the intersection of the edge joining $(\rho, 0)$ and $(\rho w, \rho w)$ of $\rho\mathbf{H}(A)$ and the edge joining $(1-\rho w, 1-\rho w)$ and $(1, 1-\rho)$ of $\rho\mathbf{H}(A)+(1,1)$. Also P is the mid-point of the edge joining the vertices $(1-\rho, 0)$ and $(1, \rho)$ of $\rho\mathbf{H}(A) + (1,0)$. The point

P' can be similarly described due to the symmetry in the line $x = y$. It is clear now that the four translates of $\rho\mathbf{H}(A)$ through the points (0, 0), (0, 1), (1, 0) and (1, 1), cover the fundamental square $0 \le x < 1, 0 \le y < 1$ and P, P' are the only just covered points in it. From the shape of $\rho\mathbf{H}(A)$ it is easily seen that P, P' are also just covered points of the configuration $(\rho\mathbf{H}(A), \mathbf{Z}^2)$. It follows that $\overline{d}(A) = \rho$ and $\mathbf{S}(A) = \{(0, \frac{1}{2}\rho, 1 - \frac{1}{2}\rho), (0, 1 - \frac{1}{2}\rho, \frac{1}{2}\rho)\}$. □

Now we prove a precise result in case either w_1 or w_2 is irrational. Theorem 3.3 follows immediately from this. This theorem can also be proved directly following the line of proofs of Theorems 3.4–3.6, keeping in mind that the projection of $\mathbf{Z}^3$ is a group under addition and contains $\mathbf{Z}^2$.

Lemma 4.1. *Let* $A = (1, w_1, w_2)$*, at least one of* w_1, w_2 *being irrational.*

(i) *If* $1, w_1, w_2$ *are linearly independent over* $\mathbf{Q}$*, then*

$$\overline{d}(A) = 0.$$

(ii) *If* $1, w_1, w_2$ *are linearly dependent over* $\mathbf{Q}$*, let* $b_0 + b_1 w_1 + b_2 w_2 = 0$, $b_0, b_1, b_2 \in \mathbf{Z}$ *and* $gcd(b_0, b_1, b_2) = 1$*. Then*

$$\overline{d}(A) = \frac{1}{2\max(|b_0|, |b_1|, |b_2|)}.$$

Proof. Let $\mathbf{M}(A)$ be the smallest rational subspace of $\mathfrak{R}^3$ containing A i.e. $\mathbf{M}(A)$ is the smallest subspace of $\mathfrak{R}^3$, generated by points of $\mathbf{Q}^3$, which contains A. Dumir, Hans-Gill and Wilker [1], proved that

$$\overline{d}(A) = \sup_{P \in \mathfrak{R}^3} d(\mathbf{Z}^3, \mathbf{M}(A) + P). \tag{4.1}$$

In case (i), $\mathbf{M}(A) = \mathfrak{R}^3$ and therefore $\overline{d}(A) = 0$.

In case (ii), $\mathbf{M}(A)$ = the subspace orthogonal to the line joining O to (b_1, b_2, b_3). It follows from (4.1) and Lemma 2 of Dumir et. al [1] that in this case

$$\overline{d}(A) = \frac{1}{2\max(|b_0|, |b_1|, |b_2|)}.$$

□

In Theorems 3.4–3.6, we shall be concerned with covering by $\phi_A(\frac{1}{2}\mathrm{C})$ which is the convex cover of $\pm(\frac{1}{2}, 0)$, $\pm(0, \frac{1}{2})$ and $\pm(\frac{1}{2}w_1, \frac{1}{2}w_2)$. It contains the square $\mathbf{S}$ with vertices $\pm(\frac{1}{2}, 0)$, and $\pm(0, \frac{1}{2})$ which is also given by the inequalities

$$|x + y| \le \tfrac{1}{2}, \quad |x - y| \le \tfrac{1}{2}.$$

Let $\mathbf{K}$ be the strip $|x - y| \le \frac{1}{2}$. We collect some simple results that we need.

Lemma 4.2. *Let $m \geq 2$ be an integer.*

(i) *The sets* $\mathbf{S} + (\frac{r}{m}, \frac{r}{m}), r \in \mathbf{Z}$, *cover the strip* $\mathbf{K}$.
If $m = 2$, *then this is a tiling.*
If $m > 2$, *then the just covered points are precisely the points on the boundary of* $\mathbf{K}$ *i.e. points on the lines* $x - y = \pm\frac{1}{2}$.

(ii) *Let P be a point of the strip* $\mathbf{K}$ *lying outside* $\mathbf{S}$. *Let* $\mathbf{H}$ *be the convex cover of* $\mathbf{S}$ *and* $\pm P$. *Then the sets* $\mathbf{H} + (\frac{r}{m}, \frac{r}{m}), r \in \mathbf{Z}$, *cover the strip* $\mathbf{K}$. *The just covered points are points on the boundary of* $\mathbf{K}$ *i.e. points on the lines* $x - y = \pm\frac{1}{2}$.

(iii) *Let* $\mathbf{P}$ *be the parallelogram with vertices* $\pm(\frac{1}{2}, 0)$, *and* $\pm(\frac{1}{2}, 1)$. *The sets* $\mathbf{P} + (r, r), r \in \mathbf{Z}$ *give tiling of* $\mathbf{K}$.

(iv) *The strips* $\mathbf{K} + (m, 0), m \in \mathbf{Z}$, *give a tiling of the plane.*

We omit the easy proof of this lemma.

Proof of Theorem 3.4. Let $w = \frac{a}{m}$, a, m positive integers, $(a, m) = 1$. In the first case $m > 1$ and

$$\phi_A(\mathbf{Z}^3) = \{(x_2 - wx_1, x_3 - wx_1) : x_1, x_2, x_3 \in \mathbf{Z}\}.$$

Since $(a, m) = 1$, there exist integers x_1, x_2 such that $mx_2 - ax_1 = 1$. Taking $x_3 = x_2$, we see that $(\frac{1}{m}, \frac{1}{m})$ is in $\phi_A(\mathbf{Z}^3)$. Since

$$\begin{aligned}(x_2 - wx_1, x_3 - wx_1) &= (x_2 - x_3, 0) + (x_3 - wx_1, x_3 - wx_1)\\ &= (x_2 - x_3, 0) + (mx_3 - ax_1)(\tfrac{1}{m}, \tfrac{1}{m})\\ &= \text{integral combination of } (1, 0) \text{ and } (\tfrac{1}{m}, \tfrac{1}{m})\end{aligned}$$

Thus $(1, 0)$ and $(\frac{1}{m}, \frac{1}{m})$ is a basis of $\phi_A(\mathbf{Z}^3)$.
In the second case $m \geq 1$ and

$$\begin{aligned}\phi_A(\mathbf{Z}^3) &= \{(x_2 - wx_1, x_3 - (1 + w)x_1) : x_1, x_2, x_3 \in \mathbf{Z}\}\\ &= \{(x_2 - wx_1, x_3 - x_1 - wx_1) : x_1, x_2, x_3 \in \mathbf{Z}\}\\ &= \{(x_2 - wx_1, x_3 - wx_1) : x_1, x_2, x_3 \in \mathbf{Z}\}\end{aligned}$$

after renaming $x_3 - x_1$ as x_3.
So it is also generated by $(1, 0)$ and $(\frac{1}{m}, \frac{1}{m})$. In the first case $\frac{1}{2}\mathbf{H}(A) = \phi_A(\frac{1}{2}\mathbf{C})$ is a hexagon with vertices $\pm(\frac{1}{2}, 0)$, $\pm(0, \frac{1}{2})$, $\pm(\frac{1}{2}w, \frac{1}{2}w)$.

In the second case, since the point (0,1/2) lies on the line segment joining $(-\frac{1}{2}, 0)$ and $(\frac{1}{2}w, \frac{1}{2}(1 + w))$, $\frac{1}{2}\mathbf{H}(A)$ a parallelogram with vertices $\pm(\frac{1}{2}, 0)$ and $\pm(\frac{1}{2}w, \frac{1}{2}(1 + w))$, In both cases $\frac{1}{2}\mathbf{H}(\mathbf{A})$ is the convex cover of **S** and a point of **K** outside **S**. Therefore for $m \geq 2$, the result follows from Lemma 4.2 in both cases. For $m = 1$, w is an integer. So we are in Case (ii). Here $\phi_A(\frac{1}{2}\mathbf{C})$ contains the parallelogram **P** with vertices $\pm(0, \frac{1}{2})$ and $\pm(\frac{1}{2}, 1)$. It also contains the open vertical sides of **P** in its interior. The result again follows from Lemma 4.2.

The set $\mathbf{S}(A)$ in the theorem corresponds to the just covered point in the fundamental parallelogram corresponding to the basis $(1, 0)$ and $(\frac{1}{m}, \frac{1}{m})$. These are the points on the boundary $x = y + \frac{1}{2}$ of **K**. □

Proof of Theorem 3.5. Here we find it convenient to divide by w and take the direction ratios as $(w', w', 1)$, $w' = \frac{1}{w} < 1$. We project Z_3 parallel to this direction on the plane $x_3 = 0$. As in Theorem 3.4, the projection of Z^3 is a lattice generated by $(1, 0)$ and $(\frac{1}{m}, \frac{1}{m})$.

The projection of $\frac{1}{2}\mathbf{C}$ is the convex cover of the square **S** and points $\pm P = \pm(\frac{w'}{2}, \frac{w'}{2})$. If $w' > \frac{1}{2}$, the points $\pm P$ lie outside **S**. Since $\pm P$ lie in the strip **K**, result follows from Lemma 4.2. If $w' \leq \frac{1}{2}$, then $\pm P$ lie in S and hence the projection of $\frac{1}{2}\mathbf{C}$ is the square **S**. Again result follows from Lemma 4.2 on distinguishing the case $w' = \frac{1}{2}$, in which case **S** gives a tiling for the projected lattice. Because of tiling in the case $w' = \frac{1}{2}$ i.e. $m = 2$, we get the translates of the line $x + y = 1/2$ as additional just covered points. □

Lemma 4.3. *Let* $\mathbf{H}_0$ *be the hexagon with vertices* $\pm(\frac{1}{2}, 0), \pm(0, \frac{1}{2}), \pm(\frac{1}{2}, \frac{1}{2})$. *Suppose that* Λ *is a lattice containing* Z^2 *having a point* $P = (p_1, p_2)$ *with* $p_1 \neq p_2, 0 \leq p_1 < 1, 0 \leq p_2 < 1$. *Then* $(\mathbf{H}_0, \Lambda)$ *is a covering, and the just covered points, if any, lie on the lattice translates of horizontal or vertical sides of* $\mathbf{H}_0$.

Proof. Since $\Lambda \supset \mathbf{Z}^2$, we can replace P by $(1, 1) - P$, if necessary and suppose that either $0 \leq p_1 \leq \frac{1}{2}$ or $0 \leq p_2 \leq \frac{1}{2}$. On renaming the coordinates, we can further suppose that $p_2 \leq p_1$. Therefore $p_2 \leq \frac{1}{2}$. If $p_2 = 0$ and $\frac{1}{2} < p_1 < 1$ then we can replace P by $(1, 0) - P$. We divide the region in which P lies into two parts:

I. $\{x(\frac{1}{2}, 0) + y(\frac{1}{2}, \frac{1}{2}) : 0 < x \leq 1, 0 \leq y \leq 1\}$ = the parallelogram generated by $(\frac{1}{2}, 0)$ and $(\frac{1}{2}, \frac{1}{2})$

II. $\{y + \frac{1}{2} \leq x < 1, 0 < y < \frac{1}{2}\}$ = the triangle with vertices $(\frac{1}{2}, 0)$, $(1, 0)$ and $(1, \frac{1}{2})$.

If P lies in I, then triangle with vertices O, P and $(1, 1)$ is covered by H_0, $\mathbf{H}_0 + P$, $\mathbf{H}_0 + (1, 1)$. If P lies in II, then the triangle with vertices O, P and $(1, 0)$ is covered by $\mathbf{H}_0$, $\mathbf{H}_0 + P$, $\mathbf{H}_0 + (1, 0)$. The statement about just covered points can also be easily verified. □

Proof of Theorem 3.6. Here $\phi_A(\frac{1}{2}\mathbf{C})$ is the convex cover of

$$\pm(\tfrac{1}{2}, 0), \pm(0, \tfrac{1}{2}), \pm(\tfrac{1}{2}w_1, \tfrac{1}{2}w_2).$$

Case i. $w_2 > w_1 + 1$

Here $\mathbf{H} = \phi_A(\frac{1}{2}\mathbf{C})$ is a parallelogram with vertices $\pm(\frac{1}{2}, 0)$ and $\pm(\frac{1}{2}w_1, \frac{1}{2}w_2)$. It contains the parallelogram $\mathbf{P}$ with vertices $\pm(\frac{1}{2}, 0)$ and $\pm(\frac{1}{2}, 1)$. Further, $\mathbf{H}$ contains the vertex $(\frac{1}{2}, 1)$ of $\mathbf{P}$ in its interior. Therefore $\mathbf{H}$ contains in its interior all points of $\mathbf{P}$ except $\pm(\frac{1}{2}, 0)$. The vertex $(\frac{1}{2}, 0)$ of $\mathbf{P}$ lies in the interior of $\mathbf{H} + (1, 1)$. Since $(\mathbf{P}, \mathbf{Z}^2)$ is a tiling (Lemma 4.2), it follows that $(\mathbf{H}, \mathbf{Z}^2)$ is a covering and there are no just covered points. Since $\phi_A(\mathbf{Z}^3)$ contains $\mathbf{Z}^2$, $(\mathbf{H}, \varphi_A(\mathbf{Z}^3))$ has the same property.

Case ii. $1 < w_1 < w_2 < w_1 + 1$.

Here $\mathbf{H} = \phi_A(\frac{1}{2}\mathbf{C})$ is a proper hexagon containing $\mathbf{H}_0$. Since $\mathbf{H}$ contains $\pm(\frac{1}{2}, \frac{1}{2})$ in its interior, it contains the vertical and horizontal sides of $\mathbf{H}_0$, other than the end points $\pm(\frac{1}{2}, 0)$ and $\pm(0, \frac{1}{2})$ in its interior.

Also, $\varphi_A(\mathbf{Z}^3)$ contains the point $P = (\{w_1\}, \{w_2\})$, where $\{w\}$ denotes the fractional part of w. Since w_1, w_2 are not both integers, $P \neq O$ and since $w_2 - w_1$ is not an integer $\{w_1\} \neq \{w_2\}$. So this lattice has a point in $0 \leq x < 1, 0 \leq y < 1$, with $x \neq y$ and it follows from Lemma 4.3 that $(\mathbf{H}, \varphi_A(\mathbf{Z}^3))$ is a covering and it is easily seen that there are no just covered points.

Since $\mathbf{H}$ is bounded, the theorem follows in both cases. □

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Hilbert's Seventeenth Problem and Pfister's Work on Quadratic Forms

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On August 8, 1900, David Hilbert [5], in his famous address at the International Congress of Mathematicians in Paris, proposed twenty three problems as sign posts for twentieth century Mathematics; the seventeenth being

Hilbert's Conjecture. A necessary and sufficient condition that $f(X_1, X_2, \dots, X_n) \in \mathbb{R}(X_1, X_2, \dots, X_n)$ is a sum of squares (sos) in $\mathbb{R}(X_1, X_2, \dots, X_n)$ is that $f(X_1, X_2, \dots, X_n)$ is, positive semi definite (psd), i.e. $f(X_1, X_2, \dots, X_n) \geq 0$ for all $a_1, a_2, \dots, a_n \in \mathbb{R}$ for which f is defined.

A similar conjecture holds for $\mathbb{Q}(X_1, X_2, \dots, X_n)$. These conjectures were proved by Artin [1] in 1927 for both $\mathbb{R}$ and $\mathbb{Q}$, but one still did not know how many squares are needed for the representation. Some results were of course known when the number of variables $n = 1$ or 2.

Indeed, for $n = 1$, Hilbert had already proved (1893), the following

Theorem 1. *Let $f(X) \in \mathbb{R}(X)$ be psd; then f(X) is a sum of two squares.*

Proof. First it is clear that we may suppose that

(i) $f(X) \in \mathbb{R}$ [X] (for otherwise $f(X) = u(X)/v(X)$, $(u(X), v(X) \in \mathbb{R}[X]) = u(X).v(X)/v^2(X)$ and we merely look at $u(X).v(X)$),

(ii) $f(X)$ is square-free.

Now factor $f(X)$ in $\mathbb{C}[X]$ to give $f(X) = \alpha(X - \alpha_1)(X - \overline{\alpha_1})\dots$. Here there are no real roots, for if $f(X) = (X - \rho)g(X)$, then for $X > \rho$, we have $g(X) \geq 0$ since f is psd, while for $X < \rho$, we have $g(X) \leq 0$, again since f is psd. It follows that $g(\rho) = 0$, contradicting the fact that f was taken square-free. Moreover, by letting $X \to \infty$, we see that $\alpha > 0$, f being psd. Thus $\alpha = b^2 (b \in \mathbb{R})$.

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Now consider the polynomial $\varphi(X) = b(X - \alpha_1)(X - \alpha_2)\ldots$, which has complex coefficients and so may be written as

$$\begin{aligned}\varphi(X) &= u(X) + \iota v(X), u(X), v(X) \in \mathbb{R}[X]. \text{Then}\\ \overline{\varphi}(X) &= u(X) - \iota v(X). \text{Multiplying, we get}\\ f(X) &= u^2(X) + v^2(X) \text{ as required.}\end{aligned}$$

□

Hilbert had also proved (1893) his conjecture for $n = 2$ for the field $\mathbb{R}(X_1, X_2, \ldots, X_n)$, i.e. the following

Theorem 2. *Let $f(X, Y) \in \mathbb{R}(X, Y)$ be psd; then $f(X, Y)$ is a sum of four squares in $\mathbb{R}(X, Y)$.*

Later (1937), another proof was given by Witt. In 1965, Pfister [10] gave a very neat proof of Theorem 2, which has the advantage that it can be generalized to the case of n variables (i.e. Theorem 4 below).

In 1966, James Ax (unpublished) proved

Theorem 3. *Let $f(X, Y, Z) \in \mathbb{R}(X, Y, Z)$ be psd; then $f(X, Y, Z)$ is a sum of eight squares in $\mathbb{R}(X, Y, Z)$.*

Pfister's general result (1967) is (see [17])

Theorem 4. *Let $f(X_1, X_2, \ldots, X_n) \in \mathbb{R}(X_1, X_2, \ldots, X_n)$ be psd; then $f(X_1, X_2, \ldots, X_n)$ is a sum of at most 2^n squares in $\mathbb{R}(X_1, X_2, \ldots, X_n)$.*

All this being a very attractive and popular topic, many survey articles have been written about it by top ranking mathematicians. Amongst the earlier ones are by Olga Tausski [21], written in 1981 and by Daniel Shapiro [20], written in 1989, while amongst the more recent ones are by Pfister [15], written in 2000 and by Schariau [19], also written in 2000. Interested readers may look these up.

The next question is "*Is the bound 2^n best possible?*" In other words, what is the true value of the **Pythagorean number** $P(\mathbb{R}(X_1, X_2, \ldots, X_n))$ of the field $\mathbb{R}(X_1, X_2, \ldots, X_n)$? This would be very difficult to answer in the general case; even to conjecture.

For $n = 1$, i.e. in $\mathbb{R}(X)$, "2" is best possible, i.e. there exist psd functions (for example $X^2 + d\ (d > 0)$) which are not squares in $\mathbb{R}(X)$.

For $n = 2$, "4" is best possible, i.e. $P(\mathbb{R}(X, Y)) = 4$. Thus we have to give an example of a psd function which cannot be expressed as a sum of three squares.

The ubiquitous polynomial $1 + X^2Y^2(X^2 - 3) + X^2Y^4$ is psd (since the arithmetic mean of 1, X^2Y^4, X^4Y^2 is at least equal to their geometric mean) but cannot be expressed as sum of three squares. This was proved by Cassels, Ellison and Pfister [2] in 1971, in a most ad hoc way using a subtle method based on elliptic curves and thus it is special to the case $n = 2$. There is a subsequent paper in the same spirit by a student of Cassels. In [3], the existence a priori of such sums of three squares is proved by a completely different method. Very recently Mahe (Rennes) and one of his students has given further examples in the style of Cassels-Ellison-Pfister.

It is easy to get the lower bound $n+1$ for the Pythagorean number $P(\mathbb{R}(X_1, X_2, \ldots, X_n))$ of $\mathbb{R}(X_1, X_2, \ldots, X_n)$, but even for $n = 3$, this only gives

$$4 \leq P(\mathbb{R}(X_1, X_2, X_3)) \leq 8$$

and the exact value is not known.

Let us now briefly go over to $\mathbb{Q}(X_1, X_2, \ldots, X_n)$.

For $n = 1$, Landau (1912), (see [17] as a more accessible reference) showed that eight squares suffice, i.e. the following

Theorem 5. *Let $f(X) \in \mathbb{Q}(X)$ be psd; then f is a sum of at most eight squares.*

It is not too difficult to show that the function x^2+7 is not a sum of four squares in $\mathbb{Q}(X)$. Thus $5 \leq P(\mathbb{Q}(X)) \leq 8$.

More generally in a field K of finite type (as a field) over $\mathbb{Q}$ let $P(d)$ be the maximum of $P(K)$ for all fields of transcendence degree d over $\mathbb{Q}$.

For $d = 0$, the results are classical; indeed by the Hasse-Minkowski theorem, $P(K) \leq 4$.

For $d = 1$ and $K = k(t)$ (a pure transcendental extension of a number field k), the bound $P(K) \leq 8$ holds (Theorem 5 above (essentially))

In 1971, Y. Pourchet [16] showed that $P(\mathbb{Q}(X)) = 5$. A comparitively elementary proof of Pourchet's result (for $\mathbb{Q}(X)$) is given in [17], pp. 241–253.

For K of transcendence degree one over $\mathbb{Q}$, $P(K) \leq 5$ but without further restriction, $P(K) \leq 7$ (see appendix of [8] – an easy consequence of a theorem of Kato). Indeed, following F. Pop (unpublished), the stronger inequality $P(1) \leq 6$ holds.

The appendix of [8] shows that for $K = F(X)$, a pure transcendental extension in one variable over a field F of functions of one variable over $\mathbb{Q}$ (for example K could be a pure transcendental extension in two variables over $\mathbb{Q}$), the upper bound $P(K) \leq 8$ holds.

Little is known regarding generalizations of these results to n $\geq$ 2. Whatever little is known to date, is summarized below:

For d $\geq$ 2, it has been proved in 1991 [4] that $P(d)$ is finite, indeed more precisely that $P(d) \geq 2^{d+1}$, subject to two conjectures:

Conjecture 1. This is the Bloch-Kato-Milnor conjecture relating Milnor K-theory mod 2, Galois cohomology with coefficients in $\mathbb{Z}/2$ and successive quotients of powers of the fundamental ideal in Witt group.

Conjecture 2. There is a local-global principle for the group of Galois cohomology $H^{d+2}(K,\mathbb{Q}/\mathbb{Z}(d+1))$ on the function fields $K/\mathbb{Q}$ with transcendence degree d (Kato conjecture).

Concerning Conjecture 1, even if all details of the proof (mainly due to Voevodsky) are not yet published, one may refer to the electronic publication [9] and to the papers which are referred there.

Concerning Conjecture 2, the local-global principle had already been announced by Jensen. The specialists seem to believe that the proof can be completed, but the paper has not been published, even after ten years. Two years ago, Arason noticed that if one wishes to prove only that $P(d) \leq 2^{d+2}$, then one may forget Conjecture 2. Hence it suffices to have only the Milnor conjecture. See pages 37, 38 of Pfister's paper [15].

Here we should like our readers to refer to a recent paper (1999) by Hoffman [6].

We now give a brief survey of Pfister's results on sums of squares. The starting point of Pfister's work on quadratic forms (of which sums of squares is a beautiful special case) was the following result of Cassels' [2]′ (see also [17]):

Theorem 6. (Cassels – 1964) *If the polynomial $f(X) \in k[X]$ is a sum of n squares of elements of $k(X)$, then it is a sum of n squares of elements of $k[X]$.*

Remark. What is new in this enunciation is that the same number n of squares suffice in $k[X]$. Without this condition, the result had been proved by Artin [1] in 1927.

We define the **Stufe (level)** $s = s(k)$ of a field k to be the least positive integer s for which the equation $-1 = \mathbf{a}_1^2 + \mathbf{a}_2^2 + \cdots + \mathbf{a}_s^2$ is solvable with $\mathbf{a}_i \in k$. If the equation has no solution, we put $s(k) = \infty$ and call k formally real. The following beautiful result is due to Pfister [11]:

Theorem 7. *For any field k, $s(k)$ (if finite) is always a power of 2. Conversely every power of 2 is the Stufe $s(k)$ of some field k.*

Intimately connected with this are the so called 2, 4 and 8 square identities:

(i) $(X_1^2 + X_2^2)(Y_1^2 + Y_2^2) = (X_1Y_1 - X_2Y_2)^2 + (X_1Y_2 + X_2Y_1)^2$,

(ii) $(X_1^2+X_2^2+X_3^2+X_4^2)(Y_1^2+Y_2^2+Y_3^2+Y_4^2) = (X_1Y_1-X_2Y_2-X_3Y_3-X_4Y_4)^2 + (X_1Y_2 + X_2Y_1 + X_3Y_4 - X_4Y_3)^2 + (X_1Y_3 + X_3Y_1 - X_2Y_4 + X_4Y_2)^2 + (X_1Y_4 + X_4Y_1 + X_2Y_3 - X_3Y_2)^2$,

(iii)
$$\begin{aligned}
&(X_1^2 + X_2^2 + \cdots + X_8^2)(Y_1^2 + Y_2^2 + \cdots + Y_8^2)\\
&= (X_1Y_1 - X_2Y_2 - X_3Y_3 - X_4Y_4 - X_5Y_5 - X_6Y_6 - X_7Y_7 - X_8Y_8)^2\\
&\quad + (X_1Y_2 + X_2Y_1 + X_3Y_4 - X_4Y_3 + X_5Y_6 - X_6Y_5 - X_7Y_8 + X_8Y_7)^2\\
&\quad + (X_1Y_3 + X_3Y_1 - X_2Y_4 + X_4Y_2 + X_5Y_7 - X_7Y_5 + X_6Y_8 - X_8Y_6)^2\\
&\quad + (X_1Y_4 + X_4Y_1 + X_2Y_3 - X_3Y_2 + X_5Y_8 - X_8Y_5 - X_6Y_7 + X_7Y_6)^2\\
&\quad + (X_1Y_5 + X_5Y_1 - X_2Y_6 + X_6Y_2 - X_3Y_7 + X_7Y_3 - X_4Y_8 + X_8Y_4)^2\\
&\quad + (X_1Y_6 + X_6Y_1 + X_2Y_5 - X_5Y_2 - X_3Y_8 + X_8Y_3 + X_4Y_7 - X_7Y_4)^2\\
&\quad + (X_1Y_7 + X_7Y_1 + X_2Y_8 - X_8Y_2 + X_3Y_5 - X_5Y_3 - X_4Y_6 + X_6Y_4)^2\\
&\quad + (X_1Y_8 + X_8Y_1 - X_2Y_7 + X_7Y_2 + X_3Y_6 - X_6Y_3 + X_4Y_5 - X_5Y_4)^2.
\end{aligned}$$

These identities emerge naturally from respectively the complex numbers, the quaternions (of Hamilton) and the octonions (Cayley numbers) and Degan, about 1822, stated (erroneously of course) that there is a like identity for 2^n squares. A great many unsuccessful attempts followed Degan's discovery of the 8-identity above to extend these formulae to the 16 case and many, realizing the impossibility of such an extension, tried to give convincing arguments to prove the impossibility. Finally in 1898, Hurwitz [7] gave a proof of the impossibility. A beautiful elementary exposition of Hurwitz' result has been given by Dickson in 1919 in Annals of Mathematics. See also [17] in this context.

It was thus totally unexpected, when in 1967, Albrecht Pfister [10] proved the following remarkable

Theorem 8. *Let k be a field and let* $n = 2^m$ *be a power of* 2. *Then there are identities*

$$(*) \quad (X_1^2 + X_2^2 + \cdots + X_n^2)(Y_1^2 + Y_2^2 + \cdots + Y_n^2) = (Z_1^2 + Z_2^2 + \cdots + Z_n^2)$$

where the Z_k *are linear functions of the* Y_j *with coefficients in* $k(X_1, X_2, \ldots, X_n)$:

$$Z_k = \sum_{i=1}^{n} T_{ki} Y_i \text{ with } T_{ki} \in k(X_1, X_2, \ldots, X_n).$$

Conversely, suppose n is not a power of 2. *Then, there is a field k such that there is no identity* (*) *with* $Z_k \in k(X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n)$. *Here, the* Z_k *are not even demanded to be linear in the* Y_j.

For elementary proofs of all the above results and much more, see [17]. It is noteworthy to realize that Pfister's proof of Hilbert's seventeenth problem (Theorem 4 above) uses the notion of Stufe, as also does his proof of Theorem 5.

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Certain Representations of Mock-Theta Functions

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1. Introduction

Recently, Denis [4], making use of the identity

$$e_q(xt - xyt)e_q(t - tx) = e_q(t - xyt) \tag{1}$$

where $e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[q]_n}$ with $[q]_n = (1-q)(1-q^2)\dots(1-q^n)$, for $n \geq 1$, $[q]_0 = 1$, established the following result:

$$\begin{aligned}\sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{x^m[(a)]_m}{[(b)]_m} {}_{A+1}\Phi_B &\left[\begin{matrix} q^{-n+m}, & (a)q^m; xq^{n-m} \\ & (b)q^m \end{matrix} \right] \\ \times {}_{C+1}\Phi_D &\left[\begin{matrix} q^{-m}, & (c); yq^m \\ & (d) \end{matrix} \right] \\ = {}_{A+C+1}\Phi_{B+D} &\left[\begin{matrix} q^{-n}, & (a), (c); xyq^n \\ & (b), (d) \end{matrix} \right].\end{aligned} \tag{2}$$

with $A = B$, $C = D$ and $\left[\begin{smallmatrix} n \\ m \end{smallmatrix} \right]$ is the q-binomial coefficient, defined by $[q]_n / [q]_m [q]_{n-m}$ and the ϕ-functions are the usual basic hypergeometric functions (cf. Section 2 for detailed definitions). The parameters of the type $(a_m), m \in N$ in small brackets shall stand for the sequence of m parameters $a_1, a_2, \dots, a_m$. If $m = A$, we shall denote it by (a) instead of (a_A).

In this paper, making use of (1.2), an attempt has been made to obtain transformation formulae for partial mock-theta as well as for mock-theta functions. Transformation formulae for mock-theta functions of order three established in this paper, had been obtained by N.J. Fine [5] by making use of different transformation formulae for the function $F(a, b; t; q)$ which is defined as:

$$F(a, b; t; q) = \sum_{n=0}^{\infty} \frac{[aq]_n}{[bq]_n} t^n. \tag{3}$$

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Our aim is to show that one can easily establish these transformation formulae for mock-theta functions of order three (earlier established by Fine) from a single result (1.2) by specializing the parameters. We shall also show that with the help of (1.2), double series-representations for mock-theta functions of order five and seven can be obtained in a very closed form.

2. Notations and Definitions

For real or complex $q\,(|q| < 1)$, put

$$(4) \qquad [\lambda]_\infty \equiv [\lambda; q]_\infty = \prod_{j=0}^{\infty}(1 - \lambda q^j)$$

and let $(\lambda; q)_\mu$ be defined by

$$(5) \qquad [\lambda; q]_\mu = \frac{[\lambda; q]_\infty}{[\lambda q^\mu; q]_\infty}$$

for arbitrary parameters λ and μ, so that

$$(6) \qquad \begin{aligned}[\lambda]_n &\equiv [\lambda; q]_n \\ &= \begin{cases} 1, & (n = 0) \\ (1-\lambda)(1-\lambda q)\ldots(1-\lambda q^{n-1}), & (n \in N = \{1, 2, \ldots\}). \end{cases}\end{aligned}$$

In case λ is an integer, say k, we denote it by q^k instead of k.

A generalized basic (or q-) hypergeometric function is defined by

$$(7) \qquad {}_A\Phi_B\left[\begin{matrix}(a); z \\ (b); q^i\end{matrix}\right] = \sum_{n=0}^{\infty} q^{i(n(n-1))/2} \frac{\prod_{j=1}^{A}[a_j; q]_n}{\prod_{j=1}^{B}[b_j; q]_n} \frac{z^n}{[q; q]_n}.$$

where, for convergence

$$|q| < 1, \text{ and } |z| < \infty \text{ when } i \in N$$

or

$$\max. \{|q|, |z|\} < 1, \text{ when } i = 0.$$

provided that no zeros appear in the denominator.

The notation ${}_A\phi_B\left[\begin{matrix}(a); z \\ (b); q^i\end{matrix}\right]_N$ shall mean the first $(N + 1)$ terms of the series defined by (2.7).

In the special case when $i = 0$, the left hand side of (2.7) is written, simply, as:

$${}_A\Phi_B\left[\begin{matrix}(a); z \\ (b)\end{matrix}\right].$$

In what follows the other notations carry their usual meaning. For the definitions of mock-theta functions of order three, five and seven one is referred to the book "Resonance of Ramanujan's Mathematics, Vol. II" by Agarwal, R.P. [1].

3. In this section we shall establish transformation formulae for mock-theta functions.

Putting $C = D = 0$, $y = \beta$, $A + 1$ for A, $B + 1$ for B, $b_{B+1} = q^{-n}$ and xq^{-n} for x in (1.2) we find:

$$\sum_{m=0}^{n}\sum_{t=0}^{n-m} \frac{[(a_{A+1})]_{m+t}[\beta]_m(-)^m x^{m+t} q^{-mt}}{[q]_m[q]_t[(b)]_{m+t} q^{m(m-1)/2}} = \sum_{r=0}^{n} \frac{[(a_{A+1})]_r (x\beta)^r}{[q]_r[(b)]_r}. \tag{8}$$

Again, taking $a_{A+1} = \alpha$, x/α for x in (3.8) and then $\alpha \to \infty$ gives:

$$\sum_{m=0}^{n}\sum_{t=0}^{n-m} \frac{[(a)]_{m+t}[\beta]_m(-)^t x^{m+t} q^{t(t-1)/2}}{[q]_m[q]_t[(b)]_{m+t}} = \sum_{r=0}^{n} \frac{[(a)]_r(-)^r (x\beta)^r q^{r(r-1)/2}}{[q]_r[(b)]_r}. \tag{9}$$

Taking $C = D = 1$, $c_1 = \alpha, d_1 = \beta$, β/α for y and then $\alpha \to \infty$ in (1.2), we get

$$\sum_{m=0}^{n}\sum_{t=0}^{n-m} \frac{[(a)]_{m+t}[q^{-n}]_{m+t}(-)^m x^{m+t} q^{(n-m)t+mn}}{[q]_m[q]_t[\beta]_m[(b)]_{m+t} q^{m(m-1)/2}} = \sum_{r=0}^{n} \frac{[q^{-n}]_r[(a)]_r(-x\beta q^n)^r q^{r(r-1)/2}}{[q]_r[(b)]_r[\beta]_r}. \tag{10}$$

Now, taking $A + 1$ for A, $B + 1$ for B, $b_{B+1} = q^{-n}$, $a_{A+1} = \alpha$, xq^{-n}/α for x in (3.10), then $\alpha \to \infty$ gives:

$$\sum_{m=0}^{n}\sum_{t=0}^{n-m} \frac{[(a)]_{m+t}(-)^t x^{m+t} q^{t(t-1)/2}}{[q]_m[q]_t[\beta]_m[(b)]_{m+t}} = \sum_{r=0}^{n} \frac{[(a)]_r (x\beta)^r q^{r(r-1)}}{[q]_r[\beta]_r[(b)]_r}. \tag{11}$$

Mock-Theta Functions of Order Three

Taking $A = B = 1$, $a_1 = q$, $b_1 = -q$, $\beta = -q$ and $x = -1$ in (3.11) we obtain:

$$f_n(q) = \sum_{r=0}^{n} \frac{q^{r^2}}{[-q]_r^2} = \sum_{m=0}^{n} \frac{(-)^m}{[-q]_m^2}\, {}_1\Phi_1 \left[\begin{matrix} q^{m+1} & ;\, 1 \\ -q^{m+1}; & q \end{matrix}\right]_{n-m} \tag{12}$$

where $f_n(q)$ is a partial mock-theta function of order three. Similar notation will be adopted onward for other partial mock-theta functions.

For $n \to \infty$, (3.12) yields

$$\begin{aligned} f(q) &= \sum_{r=0}^{\infty} \frac{q^{r^2}}{[-q]_r^2} = 2\sum_{m=0}^{\infty} \frac{(-)^m}{[-q]_m} = 2{}_2\Phi_1 \left[\begin{matrix} q, 0; -1 \\ -q \end{matrix}\right] \\ &= 2 - {}_2\Phi_1 \left[\begin{matrix} q, 0; -q \\ -q \end{matrix}\right]. \end{aligned} \tag{13}$$

(by making use of [5; (2.4) p. 2; see also [1; Chapter 2 (6.2) p. 66])

Taking $A = 1$, $B = 1$, $a_1 = q$, $b_1 = iq$, $\beta = -iq$ and $x = i$ in (3.11) we find:

$$\begin{aligned} \phi_n(q) &= \sum_{r=0}^{n} \frac{q^{r^2}}{[-q^2; q^2]_r} \\ &= \sum_{m=0}^{n} \frac{(i)^m}{[-q^2; q^2]_m}\, {}_1\Phi_1 \left[\begin{matrix} q^{m+1}; -i \\ iq^{m+1}; q \end{matrix}\right]_{n-m} \end{aligned} \tag{14}$$

where the terms, like $[\alpha; q^2]$ indicate that the parameter α is on base q^2.

Taking $n \to \infty$ and then summing the inner ${}_1\Phi_1$-series on the right side of (3.14) we get:

$$\phi(q) = \sum_{r=0}^{\infty} \frac{q^{r^2}}{[-q^2; q^2]_r} = (1-i){}_2\Phi_1 \left[\begin{matrix} q, 0; i \\ -iq \end{matrix}\right]. \tag{15}$$

Putting q^2 for q and then taking $A = B = 1$, $a_1 = q^2$, $b_1 = q^3$, $\beta = -q^2$ and $x = q$ in (3.9) we get:

$$\sum_{r=0}^{n} \frac{q^{r^2+2r}}{[q^3; q^2]_r} = \sum_{m=0}^{n} \frac{[-q^2; q^2]_m q^m}{[q^3; q^2]_m}\, {}_1\Phi_1 \left[\begin{matrix} q^{2m+2}; q^2; -q \\ q^{2m+3}; q^2 \end{matrix}\right]_{n-m}.$$

which can also be written as

$$\begin{aligned} &\sum_{r=0}^{n} \frac{q^{(r+1)^2}}{[q; q^2]_{r+1}} \\ &= \frac{q}{(1-q)} \sum_{m=0}^{n} \frac{[-q^2; q^2]_m q^m}{[q^3; q^2]_m}\, {}_1\Phi_1 \left[\begin{matrix} q^{2m+2}; q^2; -q \\ q^{2m+3}; q^2 \end{matrix}\right]_{n-m}. \end{aligned} \tag{16}$$

the notation ${}_r\phi_s\left[\begin{matrix}(a_r); q^{\mu}; x\\ (b_s); q^{\lambda}\end{matrix}\right]$ indicates that the base of the parameters in ϕ-series is q^{μ}.

Now, putting $r-1$ for r in (3.16) we get:

$$\begin{aligned}\psi_n(q) &= \sum_{r=1}^{n+1} \frac{q^{r^2}}{|q; q^2|_r}\\ &= \frac{q}{(1-q)} \sum_{m=0}^{n} \frac{[-q^2; q^2]_m q^m}{[q^3; q^2]_m} {}_1\Phi_1\left[\begin{matrix} q^{2m+2}; q^2; -q \\ q^{2m+3}; q^2 \end{matrix}\right]_{n-m}.\end{aligned}$$

which, for taking $n \to \infty$ and then summing the inner ${}_1\Phi_1$-series on the right hand side yields:

$$\psi(q) = q{}_2\Phi_1\left[\begin{matrix} q^2, -q^2; q^2; q \\ 0 \end{matrix}\right]. \tag{17}$$

Putting $A = B = 1$, $a_1 = q$, $b_1 = -\omega^2 q$, $\beta = -\omega q$ and $x = -\omega^2$ (ω being the cube root of unity) in (3.11) we obtain:

$$\begin{aligned}\chi_n(q) &= \sum_{r=0}^{n} \frac{q^{r^2}}{(1-q+q^2)\ldots(1-q^r+q^{2r})}\\ &= \sum_{m=0}^{n} \frac{(-)^m \omega^{2m}}{[-\omega q]_m[-\omega^2 q]_m} {}_1\Phi_1\left[\begin{matrix} q^{m+1}; \omega^2 \\ -\omega^2 q^{m+1}; q \end{matrix}\right]_{n-m}.\end{aligned} \tag{18}$$

As $n \to \infty$, (3.18) yields

$$\begin{aligned}\chi(q) &= \sum_{r=0}^{n} \frac{q^{r^2}}{(1-q+q^2)\ldots(1-q^r+q^{2r})}\\ &= (1+\omega^2){}_2\Phi_1\left[\begin{matrix} q, 0; -\omega^2 \\ -\omega q \end{matrix}\right].\end{aligned} \tag{19}$$

Taking $A = B = 1$, $a_1 = q$, $b_1 = q^{3/2}$, $\beta = q^{3/2}$, $x = q^{1/2}$ in (3.11) and then replacing q by q^2 we find:

$$\begin{aligned}\omega_n(q) &= \sum_{r=0}^{n} \frac{q^{2r(r+1)}}{[q; q^2]_{r+1}^2}\\ &= \frac{1}{(1-q)^2} \sum_{m=0}^{n} \frac{q^m}{[q^3; q^2]_m} {}_1\Phi_1\left[\begin{matrix} q^{2m+2}; q^2; -q \\ q^{2m+3}; q^2 \end{matrix}\right]_{n-m}.\end{aligned} \tag{20}$$

As $n \to \infty$, (3.20) yields

$$\omega(q) = \sum_{r=0}^{\infty} \frac{q^{2r(r+1)}}{[q; q^2]_{r+1}^2} = \frac{1}{(1-q)}{}_2\Phi_1\left[\begin{matrix} q^2, 0; q^2; q \\ q^3 \end{matrix}\right]. \tag{21}$$

Putting $A = 1,\ a_1 = q,\ B = 1$ and then replacing q by q^2 in (3.9) we get:

$$\begin{aligned}
&\sum_{r=0}^{n} \frac{(-)^r (x\beta)^r q^{r(r-1)}}{[b_1; q^2]_r} \\
(22)\qquad &= \sum_{m=0}^{n} \sum_{t=0}^{n-m} \frac{[q^2; q^2]_{m+t}[\beta; q^2]_m (-)^t q^{t(t-1)} x^{m+t}}{[q^2; q^2]_m [q^2; q^2]_t [b_1; q^2]_{m+t}}.
\end{aligned}$$

Now, putting $b_1 = -q^3$, $\beta = q$ and $x = -q$ in (3.22) we obtain:

$$\begin{aligned}
\nu_n(q) &= \sum_{r=0}^{n} \frac{q^{r(r+1)}}{[-q; q^2]^2_{r+1}} \\
(23)\qquad &= \frac{1}{1+q} \sum_{m=0}^{n} \frac{[q; q^2]_m (-)^m q^m}{[-q^3; q^2]_m} {}_1\Phi_1 \left[\begin{matrix} q^{2m+2}; q^2; -q \\ q^{2m+3}; q^2 \end{matrix} \right]_{n-m}.
\end{aligned}$$

As $n \to \infty$, (3.23) gives:

$$(24)\qquad \nu(q) = \sum_{r=0}^{\infty} \frac{q^{r(r+1)}}{[-q; q^2]^2_{r+1}} = {}_2\Phi_1 \left[\begin{matrix} q^2, q; q^2; -q \\ 0 \end{matrix} \right].$$

Taking $A = B = 1$, $a_1 = q$, $b_1 = \omega^2 q^{3/2}$, $\beta = \omega q^{3/2}$ and $x = \omega^2 q^{1/2}$ in (3.10) and then replacing q by q^2 we get:

$$\begin{aligned}
\rho_n(q) &= \sum_{r=0}^{n} \frac{q^{2r(r+1)}}{(1+q+q^2)\ldots(1+q^{2r+1}+q^{4r+2})} \\
&= \frac{1}{(1-\omega q)(1-\omega^2 q)} \sum_{m=0}^{n} \frac{(\omega^2 q)^m}{[\omega^2 q^3, q^2]_m [\omega q^3; q^2]_m} \\
(25)\qquad &\times {}_1\Phi_1 \left[\begin{matrix} q^{2m+2}; q^2; -\omega^2 q \\ \omega^2 q^{2m+3}; q^2 \end{matrix} \right]_{n-m}.
\end{aligned}$$

Taking $n \to \infty$ and summing the inner ${}_1\Phi_1$-series on the right side of (3.25) we get

$$\begin{aligned}
\rho(q) &= \sum_{r=0}^{n} \frac{q^{2r(r+1)}}{(1+q+q^2)\ldots(1+q^{2r+1}+q^{4r+2})} \\
(26)\qquad &= \frac{1}{(1-\omega q)} {}_2\Phi_1 \left[\begin{matrix} q^2, 0; q^2; \omega^2 q \\ \omega q^3 \end{matrix} \right].
\end{aligned}$$

where $\omega = e^{2\pi 1/3}$.

Mock-Theta Functions of Order Five
(Double Series Representation)

Putting $A = 1$, $B = 1$, $b_1 = 0$, $a_1 = q$, $\beta = -q$ and $x = -1$ in (3.11) we find:

$$f_{0,n}(q) = \sum_{r=0}^{n} \frac{q^{r^2}}{[-q]_r} = \sum_{m=0}^{n} \sum_{t=0}^{n-m} \begin{bmatrix} m+t \\ m \end{bmatrix} \frac{(-)^m q^{t(t-1)/2}}{[-q]_m}.$$

which gives $f_0(q)$ as $n \to \infty$ in the following form:

$$f_0(q) = \sum_{r=0}^{\infty} \frac{q^{r^2}}{[-q]_r} = \sum_{m,t=0}^{\infty} \begin{bmatrix} m+t \\ m \end{bmatrix} \frac{(-)^m q^{t(t-1)/2}}{[-q]_m}. \tag{27}$$

Taking $A = 4$, $a_1 = q$, $a_2 = \beta = -q$, $a_3 = iq^{1/2}$, $a_4 = -iq^{1/2}$, $B = 4$, $b_1 = b_2 = b_3 = b_4 = 0$ and $x = -1$ in (3.11) we get:

$$\begin{aligned} \phi_{0,n}(q) &= \sum_{r=0}^{n} [-q;q^2]_r q^{r^2} \\ &= \sum_{m=0}^{n} \sum_{r=0}^{n-m} \begin{bmatrix} m+t;q^2 \\ m \end{bmatrix} [-q;q^2]_{m+t} [-q;q]_t (-)^m q^{t(t-1)/2}. \end{aligned}$$

which, for $n \to \infty$ gives

$$\begin{aligned} \phi_0(q) &= \sum_{r=0}^{\infty} [-q;q^2]_r q^{r^2} \\ &= \sum_{m,t=0}^{\infty} \begin{bmatrix} m+t;q^2 \\ m \end{bmatrix} [-q;q^2]_{m+t} [-q;q]_t (-)^m q^{t(t-1)/2}. \end{aligned} \tag{28}$$

Taking $A = 1$, $a_1 = q$, $\beta = -q$, $B = 1$, $b_1 = 0$ and $x = -q$ in (3.11) we obtain:

$$\begin{aligned} f_{1,n}(q) &= \sum_{r=0}^{n} \frac{q^{r(r+1)}}{[-q]_r} \\ &= \sum_{m=0}^{n} \sum_{t=0}^{n-m} \begin{bmatrix} m+t \\ m \end{bmatrix} \frac{(-)^m q^{m+t(t+1)/2}}{[-q]_m}. \end{aligned}$$

which, for $n \to \infty$ yields

$$\begin{aligned} f_1(q) &= \sum_{r=0}^{\infty} \frac{q^{r(r+1)}}{[-q]_r} \\ &= \sum_{m,t=0}^{\infty} \begin{bmatrix} m+t \\ m \end{bmatrix} \frac{(-)^m q^{m+t(t+1)/2}}{[-q]_m}. \end{aligned} \tag{29}$$

Taking $B = 4$, $b_1 = b_2 = b_3 = b_4 = 0$, $A = 4$, $a_1 = q$, $a_2 = \beta = -q$, $a_3 = iq^{1/2}$, $a_4 = -iq^{1/2}$ and $x = -q^2$ in (3.11) we find:

$$\begin{aligned}\phi_{1,n}(q) &= \sum_{r=0}^{n}[-q;q^2]_r q^{(r+1)^2}\\ &= q\sum_{m=0}^{n}\sum_{t=0}^{n-m}\begin{bmatrix} m+t;q^2 \\ m\end{bmatrix}[-q;q^2]_{m+t}\\ &\quad \times [-q;q]_t(-q)^m q^{2m+t(t+3)/2}.\end{aligned}$$

which, for $n \to \infty$ gives

$$\begin{aligned}\phi_1(q) &= \sum_{r=0}^{\infty}[-q;q^2]_r q^{(r+1)^2}\\ &= q\sum_{m,t=0}^{\infty}\begin{bmatrix} m+t;q^2 \\ m\end{bmatrix}[-q;q^2]_{m+t}\\ &\quad \times [-q]_t(-q)^m q^{2m+t(t+3)/2}.\end{aligned}\tag{30}$$

Taking $A = 2$, $B = 2$, $b_1 = 0$, $b_2 = 0$, $a_1 = q$, $a_2 = -q$, $x = -1$, $\beta = q$ in (3.9) we get:

$$\begin{aligned}\psi_{1,n}(q) &= \sum_{r=0}^{n}[-q]_r\, q^{r(r+1)/2}\\ &= \sum_{m=0}^{n}\sum_{t=0}^{n-m}\frac{[q^2;q^2]_{m+t}(-)^m q^{t(t-1)/2}}{[q]_t}.\end{aligned}$$

which, for $n \to \infty$ gives

$$\begin{aligned}\psi_1(q) &= \sum_{r=0}^{\infty}[-q]_r q^{r(r+1)/2}\\ &= \sum_{m,t=0}^{\infty}\frac{[q^2;q^2]_{m+t}(-)^m q^{t(t-1)/2}}{[q]_t}.\end{aligned}\tag{31}$$

Again, taking $A = 2$, $a_1 = q$, $a_2 = -q$, $B = 2$, $b_1 = b_2 = 0$, $\beta = q$, $x = -q$ in (3.9) we get:

$$\begin{aligned}\psi_{0,n}(q) &= \sum_{r=0}^{n}[-q]_r\, q^{(r+1)(r+2)/2}\\ &= q\sum_{m=0}^{n}\sum_{t=0}^{n-m}\frac{[q^2;q^2]_{m+t}(-)^m q^{m+t(t+1)/2}}{[q]_t}.\end{aligned}$$

which, for $n \to \infty$ yields

$$\psi_0(q) = \sum_{r=0}^{\infty}[-q]_r q^{(r+1)(r+2)/2}$$

$$= q \sum_{m,t=0}^{\infty} \frac{[q^2; q^2]_{m+t}(-)^m q^{m+t(t+1)/2}}{[q]_t}. \tag{32}$$

Taking $A = 1, a_1 = q, B = 1, b_1 = 0, x = \beta = q^{1/2}$ and then replacing q by q^2 in (3.11) we get:

$$F_{0,n}(q) = \sum_{r=0}^{n} \frac{q^{2r^2}}{[q; q^2]_r} = \sum_{m=0}^{n}\sum_{t=0}^{n-m} \frac{[q^2; q^2]_{m+t}(-)^t q^{m+t^2}}{[q; q]_{2m}[q^2; q^2]_t},$$

which, for $n \to \infty$ gives

$$F_0(q) = \sum_{r=0}^{\infty} \frac{q^{2r^2}}{[q; q^2]_r}$$

$$= \sum_{m,t=0}^{\infty} \frac{[q^2; q^2]_{m+t}(-)^t q^{m+t^2}}{[q; q]_{2m}[q^2; q^2]_t}. \tag{33}$$

Taking $A = B = 0, x = q, y = q^{-n}$, $D + 1$ for D, $C + 1$ for C, $d_{D+1} = q^{-n}$, $c_{C+1} = q (C = D)$ in (1.2) we get:

$$\sum_{r=0}^{n} \frac{[(c)]_r}{[(d)]_r} q^r = [q]_n \sum_{m=0}^{n}\sum_{t=0}^{m} \frac{[q^{-m}]_t[(c)]_t q^{m+(m-n)t}}{[q]_m[(q)]_t[q^{-n}]_t}. \tag{34}$$

Now, taking $C = 4, c_1 = q, c_2 = c_3 = c_4 = 0, D = 4, d_1 = q^{1/2}, d_2 = -q^{1/2}$, $d_3 = q, d_4 = -q$ in (3.34), we obtain:

$$\chi_{0,n}(q) = \sum_{r=0}^{n} \frac{q^r}{[q^{r+1}]_r}$$

$$= [q]_n \sum_{m=0}^{n}\sum_{t=0}^{m} \frac{[q^{-m}]_t[q]_t q^{m+(m-n)t}}{[q]_m[q]_{2t}[q^{-n}]_t},$$

which, for $n \to \infty$ gives

$$\chi_0(q) = \sum_{r=0}^{\infty} \frac{q^r}{[q^{r+1}]_r}$$

$$= [q]_\infty \sum_{m=0}^{\infty}\sum_{t=0}^{m} \frac{[q^{-m}]_t q^{m+mt}(-)^t}{[q]_m[q^{1+t}]_t q^{t(t-1)/2}}. \tag{35}$$

Again, taking $C = 4, c_1 = q, c_2 = c_3 = c_4 = 0, D = 4, d_1 = q, d_2 = -q, d_3 = q^{3/2}, d_4 = -q^{3/2}$ in (3.34) we get:

$$\chi_{1,n}(q) = \sum_{r=0}^{n} \frac{q^r}{[q^{r+1}]_{r+1}}$$

$$= [q]_n \sum_{m=0}^{n} \sum_{t=0}^{m} \frac{[q^{-m}]_t q^{m(1+t)-nt}}{[q]_m [q^{-n}]_t [q^{1+t}]_{t+1}},$$

which, for $n \to \infty$ yields

$$\chi_1(q) = \sum_{r=0}^{\infty} \frac{q^r}{[q^{r+1}]_{r+1}}$$

$$= [q]_\infty \sum_{m=0}^{\infty} \sum_{t=0}^{m} \frac{[q^{-m}]_t q^{m(1+t)}(-)^t}{[q]_m [q^{1+t}]_{t+1} q^{t(t-1)/2}}. \tag{36}$$

Mock-Theta Functions of Order Seven (Double Series Representation)

Taking $A = 2, a_1 = q, a_2 = 0, B = 2, b_1 = q^{1/2}, b_2 = -q^{1/2}, \beta = -q$ and $x = -1$ in (3.11) we obtain:

$$\Im_{0,n}(q) = \sum_{r=0}^{n} \frac{q^{r^2}}{[q^{r+1}]_r}$$

$$= \sum_{m=0}^{n} \sum_{t=0}^{n-m} \begin{bmatrix} m+t \\ m \end{bmatrix} \frac{(-)^m q^{t(t-1)/2}}{[-q]_m [q; q^2]_{m+t}},$$

which, for $n \to \infty$ gives

$$\Im_0(q) = \sum_{r=0}^{\infty} \frac{q^{r^2}}{[q^{r+1}]_r}$$

$$= \sum_{m,t=0}^{\infty} \begin{bmatrix} m+t \\ m \end{bmatrix} \frac{(-)^m q^{t(t-1)/2}}{[-q]_m [q; q^2]_{m+t}}. \tag{37}$$

Taking $A = 2, a_1 = q, a_2 = 0, B = 2, b_1 = q^{3/2}, b_2 = -q^{3/2}, \beta = -q$ and $x = -q^2$ in (3.11) we get,

$$\Im_{1,n}(q) = \sum_{r=0}^{n} \frac{q^{(r+1)^2}}{[q^{r+1}]_{r+1}}$$

$$= q \sum_{m=0}^{n} \sum_{t=0}^{n-m} \begin{bmatrix} m+t \\ m \end{bmatrix} \frac{(-)^m q^{2(m+t)} q^{t(t-1)/2}}{[-q]_m [q; q^2]_{m+t+1}},$$

which, for $n \to \infty$ yields

$$\begin{aligned}\Im_1(q) &= \sum_{r=0}^{\infty} \frac{q^{(r+1)^2}}{[q^{r+1}]_{r+1}} \\ &= \sum_{m,t=0}^{\infty} \begin{bmatrix} m+t \\ m \end{bmatrix} \frac{(-)^m q^{2m+(t+1)(t+2)/2}}{[-q]_m [q;q^2]_{m+t+1}}. \qquad (38)\end{aligned}$$

Finally, taking $A = 2$, $a_1 = q$, $a_2 = 0$, $B = 2$, $b_1 = q^{3/2}$, $b_2 = -q^{3/2}$, $\beta = x = -q$ in (3.11) we find:

$$\begin{aligned}\Im_{2,n}(q) &= \sum_{r=0}^{n} \frac{q^{r(r+1)}}{[q^{r+1}]_{r+1}} \\ &= \sum_{m=0}^{n} \sum_{t=0}^{n-m} \begin{bmatrix} m+t \\ m \end{bmatrix} \frac{(-)^m q^{m+t(t+1)/2}}{[-q]_m [q;q^2]_{m+t+1}}\end{aligned}$$

which, for $n \to \infty$ yields

$$\begin{aligned}\Im_2(q) &= \sum_{r=0}^{\infty} \frac{q^{r(r+1)}}{[q^{r+1}]_{r+1}} \\ &= \sum_{m,t=0}^{\infty} \begin{bmatrix} m+t \\ m \end{bmatrix} \frac{(-)^m q^{m+t(t+1)/2}}{[-q]_m [q;q^2]_{m+t+1}}. \qquad (39)\end{aligned}$$

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In a subsequent communication we propose to deal with the mock-theta functions of order six and ten (cf. Andrews and Hickerson [2] and Choi [3]).

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Bi-Graceful Graphs

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A graph G is said to be bi-graceful if both G and its line graph $L(G)$ are graceful. In this paper we study bi-graceful graphs and prove that C_4 with two disjoint paths of the same length attached at any two adjacent points is bi-graceful.

1. Introduction

A graceful numbering for a graph with n vertices and e edges is a one-to-one assignment of a subset of the numbers $\{0, 1, 2, \ldots, e\}$ to the n vertices in such a way that the edges receive all the numbers from 1 to e, where the number of an edge is computed to be the absolute value of the difference between the vertex numbers at its end points.

A graph that has a graceful numbering is called a graceful graph.

2. Definition

A graph G is said to be bi-graceful if both G and its line graph $L(G)$ are graceful.

Now, we exhibit some bi-graceful graphs with their graceful numberings.

3. Results

Result 3.1. All paths are bi-graceful.

Proof. $L(P_K) = P_{K-1}$ and we know that P_K is graceful for any $K \geq 1$ [4] Hence, all non-trivial paths are bi-graceful. □

Result 3.2. C_n is bi-graceful iff $n \equiv 0, 3 \pmod 4$.

Proof. $L(C_n) = C_n$ and we know that C_n is graceful iff $n \equiv 0, 3 \pmod 4$ [4] Hence, C_n is bi-graceful iff $n \equiv 0, 3 \pmod 4$. □

Result 3.3. $K_{1,n}$ is bi-graceful if $n = 1, 2, 3, 4$ and $K_{1,n}$ is not bi-graceful if $n \geq 5$.

Proof. $L(K_{1,n}) = K_n$.

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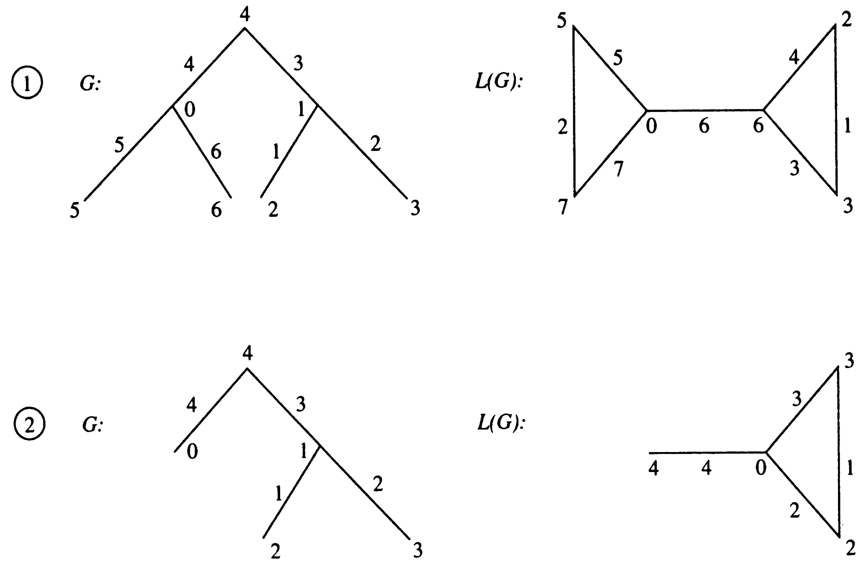


Figure 1

We know that K_n is graceful if $n = 1, 2, 3, 4$ and that K_n is not graceful if $n \geq 5$ [4]. However $K_{1,n}$ is graceful for every $n \geq 1$. Hence, $K_{1,n}$ is bi-graceful if $n = 1, 2, 3, 4$ and $K_{1,n}$ is not bi-graceful if $n \geq 5$. □

Result 3.4. If G is a (p, q) graph which is bi-graceful then $\Sigma d_i^2 \geq 4q - 2$, where $d_i = d_G(v_i)$

Proof. $L(G)$ has q points and $-q + 1/2\Sigma d_i^2$ lines.

Also we know, if a (p, q) graph is graceful then $q \geq p - 1$. So, $-q + 1/2\Sigma d_i^2 \geq q - 1$

$$1/2\ \Sigma\ d_i^2 \geq\ 2q - 1 \text{ i.e., } \Sigma\ d_i^2 \geq\ 4q - 2.$$

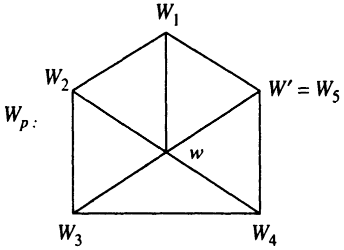
□

Result 3.5. The wheel W_p is not bi-graceful if $p = 8n$ or $8n - 2, n = 1, 2, 3, \ldots,$

Proof. We know that if $x = uv$ is a line of G then the degree of x in $L(G)$ is $d_G(u) + d_G(v) - 2$. Next we observe that if p is even, $L(W_p)$ is an Eulerian graph.

Let $x = uv$ be an edge on the cycle of W_p.

$$\begin{aligned}\text{So, the degree of } x \text{ in } L(W_p) \text{ as a point} &= d(u) + d(v) - 2\\ &= 3 + 3 - 2 = 4.\end{aligned}$$



Let $x = WW'$ be an edge of W_p, which is a spoke and let w be its centre point.

$$\text{Now}\, d(W) = p - 1.$$
$$d(W') = 3.$$

$$\begin{aligned}\text{So, the degree of } x \text{ in} L(W_p) \text{ as a point} &= d(W) + d(W') - 2\\ &= p - 1 + 3 - 2\\ &= p \quad \text{which is even.}\end{aligned}$$

Hence, $L(W_p)$ is an Eulerian graph.

Now we determine the number of lines in $L(W_p)$. We know that if G is a (p, q) graph whose points have degrees d_i, then $L(G)$ has q points and q_L lines where $q_L = -q + 1/2\Sigma d_i^2$. We know that W_p has $2(p-1)$ lines.

W_p has $p-1$ points of degree 3 and 1 point of degree $p-1$. Let q_L denote the number of lines of $L(Wp)$.

So,

$$\begin{aligned}q_L &= -q + 1/2\Sigma d_i^2 = -2(p-1) + 1/2[(p-1)3^2 + 1(p-1)^2]\\ &= [p^2 + 3p - 4]/2.\end{aligned}$$

If $p = 8n$ then,

$$q_L = [64n^2 + 24n - 4]/2 = 32n^2 + 12n - 2 \equiv 2 \,(\text{mod}\, 4).$$

If $p = 8n - 2$ then,

$$\begin{aligned}q_L &= [(8n-2)^2 + 3(8n-2) - 4]/2\\ &= [64n^2 - 8n - 6]/2 = 32n^2 - 4n - 3 \equiv 1 (\text{mod}\, 4).\end{aligned}$$

So, if $p = 8n$ or $p = 8n - 2, n = 1, 2, 3, 4, \ldots, L(W_p)$ is not graceful since, if G is graceful eulerian graph with e edges, then $e \equiv 0$ or 3 (mod 4) [4].

Hence, if $p = 8n$ or $p = 8n - 2, n = 1, 2, 3, 4, \ldots,$ the wheel W_p is not bi-graceful [4]. □

Theorem 3.6. *C_4 with two non-trivial disjoint paths of the same length, attached at any two adjacent points of C_4 is bi-graceful.*

Proof. Let the vertex set of this graph (See Fig 2) be $\{v_1, v_2, v_3, \ldots, v_n, v_{n+1}, \ldots, v_{2n}\}$, $2n \geq 6$ and edge set $\{e_1, e_2, e_3, \ldots, e_{2n-1}, e_x\}$ where e_i is incident with v_i and $v_{i+1}, i = 1, 2, 3, \ldots, 2n-1$ and e_x is incident with v_{n-1} and v_{n+2}.

Observe that $p_G = q_G = 2n$.

It's line graph is $L(G)$ (See Fig 2) with vertex set $\{e_1, e_2, e_3, \ldots, e_{2n-1}, e_x\}$ and e_i is adjacent to $e_{i+1}, i = 1, 2, 3, \ldots, 2n-2$ and e_x is adjacent with $e_{n-2}, e_{n-1}, e_{n+1}$ and e_{n+2}. Observe that $p_{L(G)} = 2n, q_{L(G)} = 2n+2$.

Now we give the graceful labeling for G by the following map.

Case 1. n is even.

$$\begin{aligned} f(v_i) &= 2n - (i-1)/2, & i &= 1, 3, \ldots, 2n-1. \\ f(v_i) &= i/2 - 1, & i &= 2, 4, \ldots, n. \\ &= i/2, & i &= n+2, n+4, \ldots, 2n. \end{aligned}$$

Case 2. *n is odd.*

$$\begin{aligned} f(v_i) &= 2n - (i-1)/2, & i &= 1, 3, \ldots, n. \\ &= 2n - 1 - (i-1)/2, & i &= n+2, \ldots, 2n-1. \\ f(v_i) &= i/2 - 1, & i &= 2, , \ldots, 2n. \end{aligned}$$

We restrict ourselves to the case when n is even. The case when n is odd is handled in a similar manner.

When i is odd, $f(v_i)$s are distinct and assume all the integers in $[n+1, 2n]$. When i is even and $n+2 \leq i \leq 2n$, $f(v_i)$s are distinct and assume all the integers in $[n/2+1, n]$.

When i is even and $2 \leq i \leq n$, $f(v_i)$s are distinct and assume all the integers in $[0, n/2-1]$. Hence, the labels of the vertex set consists of all the integers in $[0, n/2-1] \cup [n/2+1, n] \cup [n+1, 2n]$.

Hence, all the vertex labels are distinct and lie between 0 and $2n$. Now we show that all the labels from 1 to $2n$ are realized by the edges. The vertices $v_1, v_3, v_5, \ldots, v_{n-1}, v_{n+1}$ receive labels $2n, 2n-1, \ldots, 2n-n/2+1, 2n-n/2$ respectively.

The vertices $v_2, v_4, v_6, \ldots, v_n$ receive labels $0, 1, 2, \ldots, n/2-1$. Hence, the edges $v_1v_2, v_2v_3, v_3v_4, \ldots, v_nv_{n+1}$ receive labels $2n, 2n-1, 2n-2, \ldots, n+1$ respectively.

The vertices $v_{n+1}, v_{n+3}, \ldots, v_{2n-1}$ receive labels $2n-n/2, 2n-n/2-1, \ldots, n+1$ respectively.

The vertices $v_{n+2}, v_{n+4}, \ldots, v_{2n}$ receive labels $n/2+1, n/2+2, \ldots, n$ respectively. Hence, the edges $v_{n+1}v_{n+2}, v_{n+2}v_{n+3}, \ldots, v_{2n-1}v_{2n}$ receive labels $n-1, n-2, \ldots, 1$ respectively.

The label of the edge $v_{n-1}v_{n+2} = 2n - n/2 + 1 - n/2 - 1 = n$. Hence, all the labels between 1 and $2n$ are realized by the edges.

Hence, G is graceful.

Now we give the graceful labeling for $L(G)$ by the following map.

Case 1. n is even. (in $L(G)$).

$$\begin{aligned} f(e_i) &= (i-1)/2, & i &= 1,3,5,\ldots,2n-3. \\ &= n, & i &= 2n-1. \\ f(e_i) &= 2n+2-(i/2-1), & i &= 2,4,\ldots,n. \\ &= 2n-(i/2-1), & i &= n+2,\ldots,2n-2. \end{aligned}$$

and

$$f(e_x) = f(e_{n-2}) - 3.$$

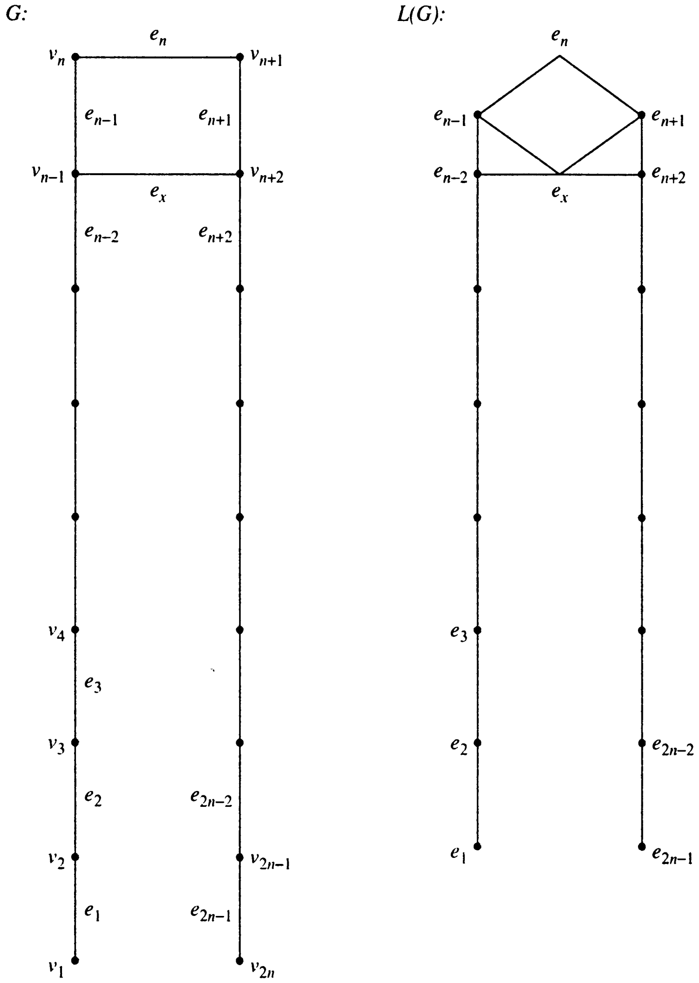


Figure 2

Case 2. n is odd. (in $L(G)$).

$$\begin{aligned} f(e_i) &= (i-1)/2, & i &= 1,3,5,\ldots,n. \\ &= (i-1)/2+2, & i &= n+2,\ldots,2n-3. \\ &= n+2, & i &= 2n-1. \\ f(e_i) &= 2n+2-(i/2-1), & i &= 2,4,\ldots,2n-2. \end{aligned}$$

and

$$f(e_x) = f(e_{n-2}) + 3.$$

We restrict ourselves to the case when n is even. The case when n is odd is handled in a similar manner.

When i is odd and $i \leq 2n-3$, $f(e_i)$ are distinct and assume all the integers in $[0, n-2]$. Also we note that $f(e_{2n-1}) = n$

When i is even and $2 \leq i \leq n$, $f(e_i)s$ are distinct and assume all the integers in $[3n/2+3, 2n+2]$.

The value of $f(e_x) = 3n/2 + 1$.

When i is even and $n+2 \leq i \leq 2n-2 f(e_i)s$ are distinct and assume all the integers in $[n+2, 3n/2]$.

Hence, the labels of the vertex set consist of all the integers in $[0, n-2] \cup \{n\} \cup [n+2, 3n/2] \cup \{3n/2+1\} \cup [3n/2+3, 2n+2]$. Hence, all the vertex labels are distinct and lie between 0 and $2n+2$.

Now we show that all the labels between 1 and $2n+2$ are realized by the edges. The vertices $e_1, e_3, \ldots, e_{n+1}$ receive labels $0, 1, 2, \ldots, n/2$ respectively. The vertices $e_2, e_4, \ldots, e_n$ receive labels $2n+2, 2n+1, \ldots, 3n/2+3$ respectively.

Hence, the edges $e_1e_2, e_2e_3, \ldots, e_ne_{n+1}$ receive labels $2n+2, 2n+1, \ldots, n+3$ respectively.

The edge label of $e_{n-1}e_x = 3n/2 + 1 - (n-2)/2 = n+2$. The edge label of $e_{n+1}e_x = 3n/2 + 1 - n/2 = n+1$. The vertices $e_{n+1}, e_{n+3}, \ldots, e_{2n-3}$ receive labels $n/2, n/2+1, \ldots, n-2$ respectively. The vertices $e_{n+2}, e_{n+4}, \ldots, e_{2n-2}$ receive labels $3n/2, 3n/2-1, \ldots, n+2$. Hence, the edges $e_{n+1}e_{n+2}$, $e_{n+2}e_{n+3}, \ldots, e_{2n-3}e_{2n-2}$ receive labels $n, n-1, \ldots, 4$ respectively.

The edge label of $e_{2n-1}e_{2n-2} = n+2-n = 2$. Since, $f(e_x) = f(e_{n-2}) - 3$, the edge label of $e_{n-2}e_x = 3$. The edge label of $e_xe_{n+2} = 3n/2 + 1 - 3n/2 = 1$.

Hence, all the labels between 1 and $2n+2$ are realized by the edges.

Hence, $L(G)$ is graceful. Hence, the theorem.

Acknowledgement

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Wheels, Cages and Cubes

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Let $G = \langle V, E\rangle$ be a graph of order $p \geq 2$ and $P = \{V_1, V_2, \ldots V_k\}$ be a partition of V of order k. The k-complement G_k^P of G is obtained as follows: For all V_i and V_j in $P, i \neq j$, remove the edges between V_i and V_j, and add the missing edges between them. G is said to be k-self-complementary if for some partition P of V of order k, $G_k^P \approx G$; and it is said to be k-co-self-complementary if $G_k^P \approx \overline{G}$. In this paper we characterize the k-self-complementary generalized wheels, cubes and cages.

1. Introduction

We consider only finite undirected graphs without loops and multiple edges. We follow the notation and terminology of [3].

Let $G = \langle V, E\rangle$ be a graph of order $p \geq 2$ and $P = \{V_1, V_2, \ldots V_k\}$ be a partition of V of order k. The k-complement G_k^P of G is obtained as follows: For all V_i and V_j in $P, i \neq j$, remove the edges between V_i and V_j, and add the missing edges between them. G is said to be k-self-complementary (k-s.c. for short) if for some partition P of V of order k, $G_k^P \approx G$ and is said to be k-co-self-complementary if $G_k^P \approx \overline{G}$.

Sampath Kumar and Pushpa Latha, in their paper [5] have discussed k-s.c. trees and unicyclic graphs. In this paper we select some classes of graphs in which almost no graph is s.c. and try to answer the question whether any graph in these classes is k-s.c. for some k. We consider the classes of generalized wheels, cubes and cages.

The generalized wheel $W_{m,n} = \overline{K}_m + C_n$ defined by Buckley and Harary [1] has m central vertices (vertices of $\overline{K}_m$) and n peripheral vertices (vertices of C_n). Every central vertex is adjacent to all the peripheral vertices. When $m = 1$, $W_{1,n} = \overline{K}_1 + C_n$.

An m-regular graph of girth n with the least possible number of vertices is called a (m, n)-cage. The $(2, n)$-cage is the n cycle, the $(m, 3)$-case is K_{m+1}, and the $(m, 4)$ cage is $K_{m,m}$. If we denote by $f(m, n)$ the number of vertices in an

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(m,n)-cage, then for $m \geq 3$,

$$f(m,n) \geq \begin{cases} \dfrac{m(m-1)^r - 2}{m-2} & \text{if } n = 2r+1 \\[2ex] \dfrac{2(m-1)^r - 2}{m-2} & \text{if } n = 2r \end{cases} \tag{1.1}$$

In [4], it has been shown by Hoffman and Singleton that, for $m \geq 3$ and $n \geq 5$, equality can hold in (1.1) only if $n = 5$ and $m = 3, 7$ or 57 or $6, 8$ or 12. We show that no (m,n)-cage other than the Petersen graph which is a $(3,5)$-cage, is k-s.c. for any $k \geq 2$. We also show that Petersen graph is only 2-s.c.

The cube Q_n is recursively defined by $Q_1 = K_2$, $Q_n = Q_{n-1} \times K_2$.

Let $G = \langle V, E\rangle$ be a graph of order $p \geq 2$ and $P = \{V_1, V_2, \ldots V_k\}$ be a partition of V of order k with $|V_i| = p_i$, $1 \geq i \leq k$. For any vertex $v \in V_i$, $1 \leq i \leq k$, the degree of v in the induced subgraph $\langle V_i\rangle$ is called the indegree of v in G, denoted by $i_G(v)$ and the difference between the degree of v in G and the indegree of v in G is called the outdegree of v, denoted by $o_G(v)$. If G is k-s.c. then

$$\sum_{\substack{j=1 \\ j\neq i}}^{k} p_j = 2O_G(v) \tag{1.2}$$

2. The Generalized Wheels

In the results of the section, we make use of the following observations which follow directly by the definition of isomorphism between two graphs.

Observation 2.1. $(W_{m,n})_k^P \neq W_{m,n}$ *if one of the following is true.*

(i) *there is a vertex v in $(W_{m,n})_k^P$ with degree not equal to n or $m+2$.*

(ii) *there is a vertex u in $(W_{m,n})_k^P$ with degree n which is non-adjacent to a vertex v in $(W_{m,n})_k^P$ of degree $(m+2)$.*

(iii) *two vertices of degree n in $(W_{m,n})_k^P$ are adjacent.*

Observation 2.2. $(W_{m,n})_k^P \neq \overline{W}_{m,n}$ *if one of the following is true.*

(i) *two vertices of degree $(m-1)$ in $(W_{m,n})_k^P$ are non-adjacent.*

(ii) *there is a vertex v in $(W_{m,n})_k^P$ with degree not equal to $(m-1)$ or $(n-3)$.*

(iii) *there is a vertex v in $(W_{m,n})_k^P$ with degree $(m-1)$ which is adjacent to a vertex u in $(W_{m,n})_k^P$ with degree $(n-3)$.*

Theorem 2.1. $(W_{m,n})_k^P \cong W_{m,n}$ *if and only if $k = 2$, $m = 1$ and $n = 5$.*

Proof. Let C denote the set of central vertices and F denote the set of peripheral vertices of $W_{m,n}$. We consider the following two cases.

Case i. Let $C \subseteq V_i$, for some i, $1 \leq i \leq k$. We observe that for any two vertices v and u in C either both of them remain central vertices or both of them become peripheral vertices in $(W_{m,n})_k^P$. No central vertex of $W_{m,n}$ can remain central vertex of $(W_{m,n})_k^P$. Hence, $(m+2)$ peripheral vertices of G must be in V_i and $n-(m+2)$ peripheral vertices must be in other parts of the partition P. No vertex in V_j, $j \neq i$ can become a central vertex of $(W_{m,n})_k^P$. Hence m out of $(m+2)$ peripheral vertices of $W_{m,n}$ in V_i must become central vertices of $(W_{m,n})_k^P$. These m vertices are of outdegree zero in $W_{m,n}$ and remaining two vertices are adjacent to all these m vertices. Hence $m = 1$ and the two vertices are of outdegree 1. Let u and v be the vertices of V_i with outdegree 1.

The vertices u and v have degree $(m+2)$ in $(W_{m,n})_k^P$ only when there is only one vertex in $W_{m,n}$ outside V_i which is adjacent to u and non-adjacent to v and only one vertex in $W_{m,n}$ outside V_i which is adjacent to v and nonadjacent to u. Thus there are exactly two peripheral vertices in $W_{m,n}$ which are outside V_i. Hence $n = 5$.

If these two peripheral vertices outside V_i are in two different parts then their degree in $(W_{m,n})_k^P$ becomes $(m+1)$. Hence both of them must be in the same part, say V_2. The wheel $W_{m,n} = W_{1,5}$ is 2-s.c. with respect to the partition explained above, as is shown in the figure below.

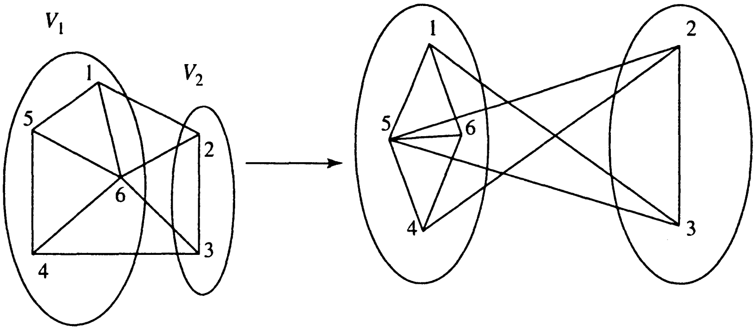


Figure 1

Case ii. Let $P = \{V_1, V_2, \ldots V_k\}$ where each V_i has m_i central vertices and n_i peripheral vertices with

$$\sum_{i=1}^{k} m_i = m \text{ and } \sum_{i=1}^{k} n_i = n.$$

If a central vertex in V_i of $W_{m,n}$ remains as a central vertex of $(W_{m,n})_k^P$ then all the central vertices of $W_{m,n}$ in V_i must remain as central vertices of $(W_{m,n})_k^P$. All the central vertices of V_j, $j \neq i$ must become peripheral vertices of $(W_{m,n})_k^P$. Peripheral vertices of $W_{m,n}$ in V_i must become peripheral vertices of $(W_{m,n})_k^P$ and peripheral vertices of $W_{m,n}$ in V_j, $j \neq i$ must become central vertices of $(W_{m,n})_k^P$, which is not possible.

Hence no central vertex of $W_{m,n}$ remains as a central vertex of $(W_{m,n})_k^P$. Let all central vertices of $W_{m,n}$ become peripheral vertices of $(W_{m,n})_k^P$. Then m of the n peripheral vertices of $W_{m,n}$ must become the central vertices of $(W_{m,n})_k^P$. Let $v \in V_i$, $1 \leq i \leq k$, be a peripheral vertex of $W_{m,n}$ which becomes a central vertex of $(W_{m,n})_k^P$. Then v cannot be adjacent to $u \in V_j$, $j \neq i$ where u is a central vertex of $W_{m,n}$ which has become a peripheral vertex of $(W_{m,n})_k^P$. Hence $(W_{m,n})_k^P$ can never be isomorphic to $(W_{m,n})$. □

Theorem 2.2. *$(W_{m,n})_k^P \cong \overline{W}_{m,n}$ if and only if $k = m + 1, n = 5$ for any integer $m \geq 1$.*

Proof. If $n = 5$ and we take each central vertex as a part and all the peripheral vertices as one part of the partition P then $(W_{m,n})_k^P \cong \overline{W}_{m,n}$ for $k = m + 1$. Now we have to prove that five is the only possible value that n can take and we show that when $m = 1$ and $n = 5$, $W_{1,5}$ has a different 2-co-s.c. partition also. To prove this we consider the following two cases.

Case i. If all the central vertices of $W_{m,n}$ are in the same part say V_i for some i, $1 \leq i \leq k$, as they are mutually non-adjacent, their degree in $(W_{m,n})_k^P$ must be $(n - 3)$. Hence $(n - 3)$ peripheral vertices of $W_{m,n}$ must be in V_i and remaining three peripheral vertices of $W_{m,n}$ must be in the other parts of P. Hence k can be at most four.

If $k = 4$ then the three other peripheral vertices are in three different parts and outdegree of each of them is three. The degree of each of them in $(W_{m,n})_k^P$ becomes $(n - 3)$. Let v be a peripheral vertex of $W_{m,n}$ in V_i, $1 \leq i \leq k$. Then degree of v in $(W_{m,n})_k^P$ is $(m + 5)$, $(m + 3)$ or $(m + 1)$ according as the outdegree of V in $W_{m,n}$ is 0, 1 or 2 respectively. Hence $(W_{m,n})_k^P \neq \overline{W}_{m,n}$.

If $k = 3$, either V_2 or V_3 has two peripheral vertices and the other one has one peripheral vertex. The following cases are possible.

In all the above cases $(W_{m,n})_k^P \neq \overline{W}_{m,n}$.

Now consider the case $k = 2$. Either V_1 or V_2 has a peripheral vertex which is adjacent to exactly one peripheral vertex in the other set. If V_1 has such a vertex then the degree of that vertex in $(W_{m,n})_2^P$ becomes $m+3$. $m+3 = n-3 \Rightarrow n = m+6$.

The degree of peripheral vertex of $W_{m,n}$ in V_2 in $(W_{m,n})_2^P$ becomes $m+3, m+1$ or $m + 5$ according as its outdegree in $W_{m,n}$ is $m + 1, m + 2$ or m respectively. Hence $(W_{m,n})_2^P \neq \overline{W}_{m,n}$.

If the above said peripheral vertex is in V_2 then its degree in $(W_{m,n})_2^P$ is $n - 3$. The other two vertices in V_2 may be of outdegree $m + 1$ and $m + 2$ or $m + 1$

and m. If other two vertices in V_2 are of outdegree $m+1$ and $m+2$ then their corresponding degrees in $(W_{m,n})_k^p$ are $n-3$ and $n-5$ respectively.

$$n-5=m-1 \Rightarrow n=m+4.$$

The degree of a peripheral vertex in V_1 in $(W_{m,n})_2^p$ becomes $n+1, n-1$ or $n-3$ according as its outdegree in $W_{m,n}$ is 2, 0 or 1 respectively. Hence no vertex in V_1 can be of degree $n-5=m-1$ in $(W_{m,n})_2^p$. Hence there is only one vertex of degree $m-1$ in $(W_{m,n})_2^p$ i.e. $m=1 \Rightarrow n=5$.

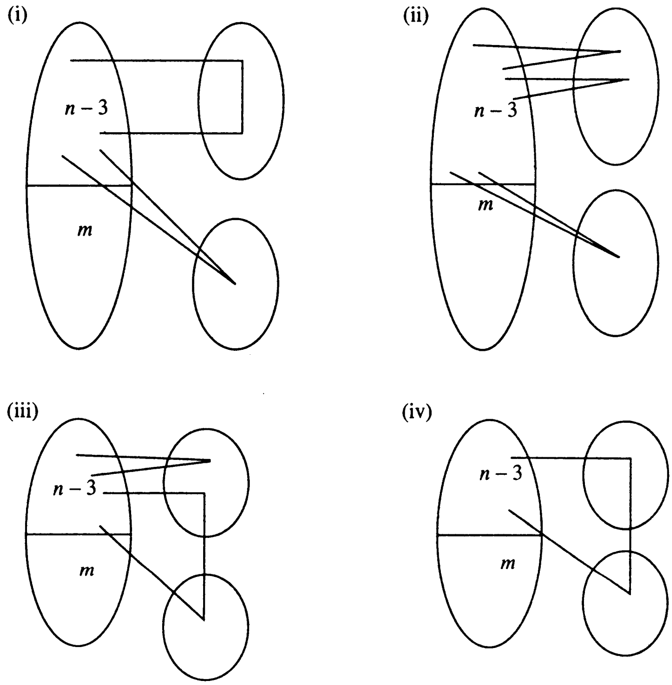


Figure 2

$W_{m,n}=W_{1,5}$ with respect to the above explained partition is 2-co s.c., as shown in the figure below,

If the other two vertices in V_2 are of outdegree 0 and 1 then the vertex of outdegree zero in $W_{m,n}$ becomes a vertex of degree $n-1$ in $(W_{m,n})_2^p$. Hence $(W_{m,n})_2^p \neq \overline{W}_{m,n}$.

Case ii. Let the central vertices be distributed among k parts of the partition P. Let V_j have m_j central vertices and n_j peripheral vertices so that $\sum_{j=1}^k m_j = m$ and $\sum_{j=1}^k n_j = n$.

Let v be a central vertex of $W_{m,n}$ and $v \in V_j$, $1 \leq j \leq k$, then degree of v in $(W_{m,n})_k^p = m - m_j + n_j$. Let u be a peripheral vertex of $W_{m,n}$ and $u \in V_j$,

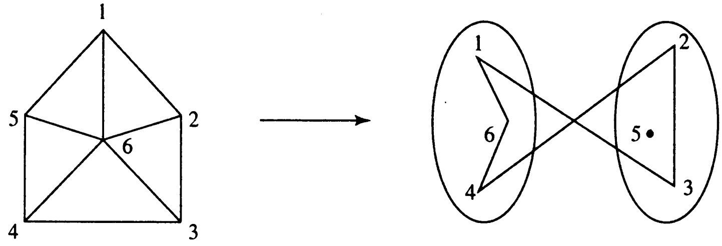


Figure 3

$1 \leq j \leq k$. Then degree of u in $(W_{m,n})_k^p$ becomes $(n+2+m_j-n_j)$, $(n+m_j-n_j)$ or $(n-2+m_j-n_j)$ according as its outdegree in $W_{m,n}$ is 0, 1 or 2 respectively. $m-m_j+n_j$ is either $m-1$ or $n-3$.

Let $v \in V_j$ be a central vertex of $W_{m,n}$. If degree of v in $(W_{m,n})_k^p = m-1$ then $m-(m_j-n_j)=m-1 \Rightarrow m_j-n_j=1$. Degree of any peripheral vertex of $W_{m,n}$ in V_j in $(W_{m,n})_k^p$ becomes $n+3$, $n+1$ or $n-1$ according as its outdegree in $W_{m,n}$ is 0, 1 or 2 respectively. Hence $(W_{m,n})_k^p \neq \overline{W}_{m,n}$.

Let the degree of every central vertex of $W_{m,n}$ in $(W_{m,n})_k^p$ be $n-3$. Then $m-n+3=m_j-n_j$ for every j.

Then degree of peripheral vertex v of $W_{m,n}$ in V_j becomes $m+5, m+3$ or $m+1$ in $(W_{m,n})_k^p$ according as its outdegree in $W_{m,n}$ is 0, 1 or 2 respectively. Hence $(W_{m,n})_k^p \neq \overline{W}_{m,n}$. □

3. Cages

If an m-regular graph G is k-s.c. then by (1.2) number of vertices in the other sets is equal to $2O_G(v)$. The maximum value that $O_G(v)$ can take is m. Hence number of vertices in the other sets must be at most $2m$. Hence total number of vertices in the graph G is at most $4m$.

Let n be odd.

$$f(m,n) \geq \frac{m(m-1)^r-2}{m-2}$$

When $n=3$, $r=1$,

$$\frac{m(m-1)^r-2}{m-2} = \frac{m(m-1)^1-2}{m-2} = m+1$$

and $m+1 < 4m \ \forall m \geq 0$.

When $n=5, r=2$

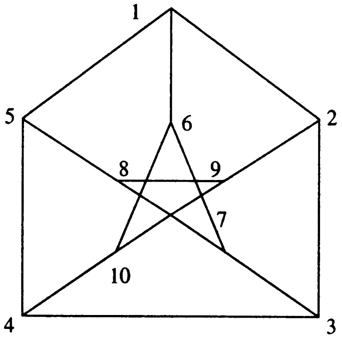
$$\frac{m(m-1)^r-2}{m-2} = \frac{m(m-1)^2-2}{m-2} = m^2+1.$$

If $m \geq 4$ then $m^2 + 1 > 4m$. Therefore, for $n \geq 5$ and $m \geq 4$, $f(m, n) > 4m$. Hence the (m, n)-cage is not k-s.c. for any $k \geq 2$.

An $(m, 3)$ cage is K_{m+1} which is not k-s.c. for any $k \geq 2$.

When $m = 3$ and $n = 5$, then $(3, 5)$-cage, which is Petersen graph, be denoted by G.

Let $P = \{V_1, V_2, \ldots V_k\}$ be a partition of $V(G)$ of order k. If there exists a vertex of degree other than three in G_k^P, then $G_k^P \neq G$. If for some i, $1 \leq i \leq k$, V_i has one vertex, two vertices or three vertices then $G_k^P \neq G$. If there are four vertices in V_i, then the remaining six vertices must be in the same part and every vertex in V_i must be of outdegree zero. Hence G is only 2-s.c. and the situation is shown below.



$P = \{V_1, V_2\}$ $V_1 = \{1, 4, 7, 9\}$ $V_2 = \{2, 3, 5, 6, 8, 10\}$

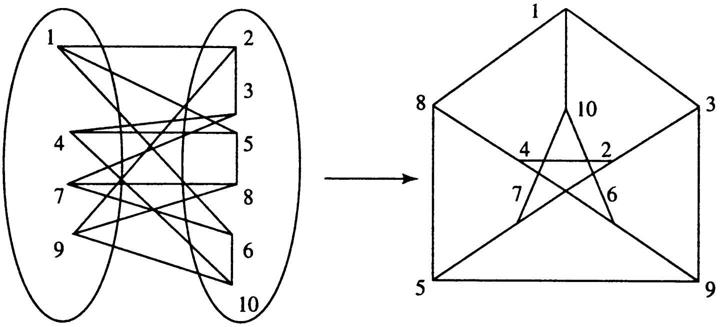


Figure 4

Let n be even. Consider $\frac{2(m-1)^r-2}{m-2}$ when $n = 4$, $r = 2$ and

$$\frac{2(m-1)^2-2}{m-2} = 2m, \; 2m < 4m \;\; \forall m.$$

When $m = 6, r = 3$ and

$$\frac{2(m-1)^3-2}{m-2} = 2(m^2-m+1),$$

$$2(m^2-m+1) > 4m \text{ for } m \geq 3.$$

Hence for $m \geq 3$ and $n \geq 6$, n even, $f(m,n) > 4m$ hence the (m,n)-cage is not k-s.c. for any $k \geq 2$. An $(m,4)$-cage is Km, m which is 2-s.c. for every m, which can be shown as follows.

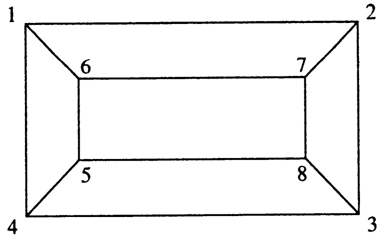
Let $V(Km, m) = \{v_1, v_2, \dots v_m, u_1, u_2, \dots u_m\}$. Let the bipartition in Km, m be $\{v_1, v_2, \dots v_m\}$ and $\{u_1, u_2, \dots u_m\}$. Consider the partition $P = \{V_1, V_2\}$ where $V_1 = \{v_1, u_1\}$ and $V_2 = V - V_1$.

With respect to the above partition $(K_{m,m})_k^P \cong K_{m,m}$.

4. The Cubes

The cube Q_n is recursively defined by $Q_1 = K_2$, $Q_n = K_2 \times Q_{n-1}$. Thus Q_n is an n-regular graph on 2^n vertices.

Remark 4.1. Q_3 is 2-s.c. as shown in the figure below.



$P = \{V_1, V_2\}$ $V_1 = \{1, 6, 3, 8\}$ $V_2 = \{2, 7, 5, 4\}$

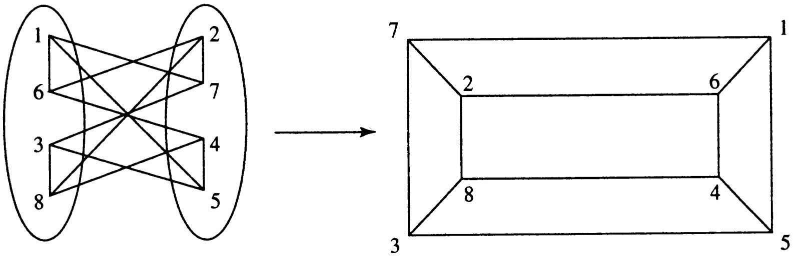


Figure 5

(2) For $n \geq 4$, $2^n > 4n$. Hence Q_n is not k-s.c. for any $k \geq 2$.

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Relevance of Srinivasa Ramanujan at the Dawn of the New Millennium

K. Srinivasa Rao

The work of Ramanujan will be appreciated, as long as people do mathematics, opined the Astrophysicist Nobel Laureate Dr. S. Chandrasekhar, at the time of the birth centenary of Ramanujan. Prof. E.H. Neville began a broadcast in Hindustani, in 1941, as follows: *Srinivasa Ramanujan was a mathematician so great that his name transcends jealousies, the one superlatively great mathematician whom India has produced in the last thousand years.* Undoubtedly, Srinivasa Ramanujan (Dec. 22, 1887 – April 26, 1920) is one of the greatest Mathematicians of the twentieth century. For his mathematical abilities and *natural genius* he has been compared, by his contemporaries, Professors G.H. Hardy and J.E. Littlewood, with all-time great mathematicians, Leonhard Euler, Carl Friedrich Gauss and Karl Gustav Jacobi. Marc Kac said: *An ordinary genius is a fellow that you and I would be just as good as, if we were only many times better. There is no mystery as to how his mind works. Once we understand what he has done, we feel certain that we too, could have done it. It is different with the magicians . . . the working of their minds is for all intents and purposes incomprehensible. Even after we understand what they have done, the process by which they have done it is completely dark.* Prof. Bruce C. Berndt who has methodically and thoroughly edited every one of the 3254 entries of Ramanujan in his three Notebooks – in *Ramanujan's Notebooks*, Parts I to V, published by Springer-Verlag (1985–1997) – states that though there are a few scattered errors in these notebooks, *Ramanujan's accuracy is amazing and mystery . . . still surrounds some of his work.* This sums up the kind of mathematical genius that Ramanujan was. The original notebooks of Ramanujan are now with the University of Madras.

Due to the orchestrated efforts of Ramanujan's friends and admirers – Professor V. Ramaswamy Iyer, Dewan Bahadur S. Ramachandra Rao, Mr. S. Narayana Iyer, Prof. P.V. Seshu Iyer, Sir Francis Spring, Prof. G.T. Walker, Prof. E.H. Neville – after a month, in 1912, as a clerk in the Accountant General's office in Madras, he secured a clerical post (Class III, Grade IV) in the Account's section of the Madras Port Trust, before recognition came to him in the form of the *first research*

2000 Mathematics Subject Classification: 00A15.

scholarship of the University of Madras in May 1913, even though he did not have the academic degrees considered essential for research. Then, due to Ramanujan's initiative – his historic first letter to Prof. Hardy – came the recognition of his talent and the invitation to go to Cambridge from Prof. Hardy, in 1914, with financial support from the University of Madras.

The authorities of the University of Madras have to be commended for their interest in nurturing Ramanujan's mathematical talent by granting him not only the first research scholarship in Mathematics of the University, for two years; but also offering him a scholarship, of £250 per year, to go to Trinity College, Cambridge, along with an adequate allowance (of £100) for his passage by ship and initial outfit; extending his scholarship until his return to India in March 1919 – on the basis of Ramanujan's prolific research output from Cambridge endorsed and strongly recommended by Prof. Hardy; and offering him an allowance, on his return, to enable him to continue his research work, until his untimely demise.

Ramanujan was awarded the B.A. degree by research, in March 1916, for his work on *Highly Composite Numbers*, his longest paper (62 pages) in the Journal of the London Mathematical Society. He was the first Indian Mathematician to be awarded the prestigious Fellowship of the Royal Society, in Feb. 1918. The citation read: *Research Student in Mathematics, Distinguished as a pure mathematician particularly for his investigations in elliptic functions and the theory of numbers*. Ramanujan was elected to a Trinity College Fellowship, in October 1918 – a Prize fellowship worth £250 a year for six years (which he was not destined to enjoy, since he returned to India in March 1919 and died in April 1920).

The publication of the *Collected Papers of Srinivasa Ramanujan* (Edited by P.V. Seshu Iyer, B.M. Wilson and G.H. Hardy, for the Cambridge University Press), in 1927 and Hardy's book entitled: *Ramanujan: Twelve lectures on subjectssuggested by his life and work* (1940), created a flurry of publications by contemporary mathematicians – Professors G.N. Watson, W.N. Bailey, C.T. Preece and others.

In 1923, Prof. Hardy spent a few months editing a chapter in the Notebooks of Ramanujan, on hypergeometric series, and found it so daunting a task that he felt that he would not be able to do any other research work if he continued to look into these Notebooks. He persuaded the University of Madras to take up this task and in 1931, Prof. G.N. Watson agreed to edit the notebooks in collaboration with Prof. B.M. Wilson. Unfortunately, the untimely death of Wilson, in 1935, at the age of 38, put an end to this effort. Prof. Watson, however, lectured on Ramanujan's Notebooks at the London Mathematical Society, in 1931 and his Presidential Address to the Royal Society, in 1935, was on 'mock' theta functions invented by Ramanujan. Twenty-two years later, in 1957, due to the efforts of Professors S.R. Ranganathan and K.S. Krishnan, Homi J. Bhabha and K. Chandrasekharan, the Tata Institute of Fundamental Research brought out a facsimile edition of these notebooks, without any commentary. This resulted in a spurt of activity on Ramanujan's entries in the Notebooks.

The discovery of the so called *'Lost' Notebook of Ramanujan* by Prof. George E. Andrews, in the spring of 1976, in the estate of late Prof. G.N. Watson, contributed to a resurgence of interest in the life and work of Ramanujan. This 'Lost' Notebook contained some 600 theorems on what Ramanujan called as 'mock' theta functions. These are results he noted on about 100 loose sheets of paper, during the last year of his life, after his return to India, in March 1919. Prof. Berndt and Prof. Andrews are at present editing this 'Lost' Notebook. A facsimile edition of the 'Lost' Notebook of Ramanujan was released by the then Prime Minister of India, Mr. Rajiv Gandhi, at an International Conference in Chennai, organized by the Institute of Mathematical Sciences to mark the birth centenary celebrations of Ramanujan, on Dec. 22, 1987. This volume contains an article by Prof. Andrews about the genesis of this 'Lost' Notebook and includes some of the Unpublished papers of Ramanujan.

It is a remarkable fact that hundreds of papers have been inspired by Ramanujan's entries in his Notebooks and his *Collected Papers*. Furthermore, Ramanujan's name has appeared in the titles and abstracts of innumerable research papers and this is continuing unabated at the dawn of the 21st century.

It is also significant to note that today there are three journals named after Srinivasa Ramanujan and these are: **The Hardy – Ramanujan Journal** (since 1975); the **Journal of the Ramanujan Mathematical Society** (since 1985); and **Ramanujan Journal** (since 1997). This is a tribute befitting the greatest.

The following is an assessment of Ramanujan, the mathematician: "Paul Erdös (himself a renowned Hungarian mathematician who created the fields of Graph Theory and Algebraic Number Theory) has passed on to us Hardy's personal ratings of mathematicians: Suppose that we rate mathematicians on the basis of pure talent on a scale from 0 to 100, Hardy gave himself a score of 25, Littlewood 30, Hilbert 80, and Ramanujan 100." (Prof. Berndt, 1985).

The *Collected Papers* and Hardy's *Ramanujan*, referred above, are two books which have been reprinted by the American Mathematical Society, in 1999, with editorial comments and notes by Prof. Berndt. The National Board for Higher Mathematics has also released, in March 2000, an Indian edition of the *Collected Papers*, without the two interesting biographical articles which were an integral part of the first edition.

A Ramanujan Museum was created in 1993 in the premises of the Avvai Academy in Royapuram by the dedicated Mathematics School Teacher Mr. P.K. Srinivasan with the help of Mr. A.T.B. Bose. In 1967, Mr. P.K. Srinivasan brought out two volumes from the Muthialpet High School to commemorate Ramanujan's 75th Birth Anniversary. He was also responsible for bringing out three booklets entitled *Creativity of Ramanujan* for Primary, Middle and Higher Secondary Schools, published by the Association for Mathematics Teachers of India. In these some of the simple entries of Ramanujan are reproduced with explanations.

The following is the Taxi cab number anecdote: Hardy started a conversation at a nursing home where Ramanujan was with the statement that the number 1729 was rather a dull number, to which Ramanujan immediately reacted by saying *No.*

***It is the smallest number expressible as the sum of two cubes in two different ways*:**

$$1729 = 1^3 + 12^3 = 9^3 + 10^3.$$

As a sequel, Ramanujan was asked by Hardy, what was the smallest number that could be expressed as the sum of two fourth powers in two different ways and Ramanujan after some thought is reported to have told Hardy that it should be a large number. Hardy later recalled that:

$$59^4 + 158^4 = 133^4 + 134^4 = 635318657$$

is the smallest number for the fourth power problem, a solution which was known to Euler. Ramanujan's quick response to the Taxi cab number is attributable to the fact that in three questions he posed to the Journal of Indian Mathematical Society (JIMS) and in two entries in his Notebooks, reference has been made to the Euler equation $X^3 + Y^3 + Z^3 = U^3$ or its solutions. Though the number 1729 itself does not find an explicit mention, an entry on p. 225, Chapter XVIII, of his second Notebook reads: $1^3 + 12^3 = 9^3 + 10^3$. The Taxi cab number anecdote could be an interesting way of introducing the subject of Diophantine equations in Higher Secondary Schools and Colleges.

The following are a few examples out of the 59 questions posed by Ramanujan, in the JIMS, which could be part of the curriculum in the appropriate stages of the mathematics education of our country:

- Q. 289, JIMS vol. IV, p. 226: Find the values of:

(i) $\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+\ldots}}}}}$

(ii) $\sqrt{6+2\sqrt{7+3\sqrt{8+4\sqrt{9+5\sqrt{10+\ldots}}}}}$

Persuaded by his friends, Ramanujan provided the answer to this question himself:

(i) Let $f(n) = n(n+2)$, then

$$\begin{aligned} f(n) &= n\sqrt{(n+2)^2} \\ &= n\sqrt{n^2+4n+4} \\ &= n\sqrt{1+(n+1)(n+3)} \\ &= n\sqrt{1+f(n+1)}. \end{aligned}$$

By recursive use of this equation, it is straightforward to show:

$$n(n+2) = n\sqrt{1+(n+1)\sqrt{1+(n+2)\sqrt{1+(n+3)\sqrt{1+(n+4)\sqrt{1+\ldots}}}}}$$

and in this if we set $n = 1$, we get the desired result that (i) in Q. 289 is nothing but 3 represented in the nested root form!

(ii) Let $f(n) = n(n+3)$, then

$$\begin{aligned} f(n) &= n\sqrt{(n+3)^2} \\ &= n\sqrt{n^2+6n+9} \\ &= n\sqrt{(n+5)+(n+1)(n+4)} \\ &= n\sqrt{(n+5)+f(n+1)}. \end{aligned}$$

By recursive use of this equation, it is straightforward to show:

$$n(n+3) = n\sqrt{(n+5)+(n+1)\sqrt{(n+6)+(n+2)\sqrt{(n+7)+(n+3)\sqrt{(n+8)+\dots}}}}$$

and in this if we set $n = 1$, we get the desired result that (ii) in Q. 289 is nothing but 4 represented in the nested root form! In this elementary mathematics example, one can see how Ramanujan should have arrived at the question itself:

$$\begin{aligned} 3 &= \sqrt{9} = \sqrt{1+8} = \sqrt{1+2*4} = \sqrt{1+2\sqrt{16}} \\ &= \sqrt{1+2\sqrt{1+15}} = \sqrt{1+2\sqrt{1+3*5}} \\ &= \sqrt{1+2\sqrt{1+3\sqrt{25}}} = \sqrt{1+2\sqrt{1+3\sqrt{1+24}}} \\ &= \sqrt{1+2\sqrt{1+3\sqrt{1+4*6}}} = \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{36}}}} \\ &= \cdots = \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+\dots}}}}} \end{aligned}$$

$$\begin{aligned} 4 &= \sqrt{16} = \sqrt{6+10} = \sqrt{6+2*5} \\ &= \sqrt{6+2\sqrt{25}} = \sqrt{6+2\sqrt{7+3*6}} \\ &= \sqrt{6+2\sqrt{7+3\sqrt{36}}} = \sqrt{6+2\sqrt{7+3\sqrt{8+28}}} \\ &= \sqrt{6+2\sqrt{7+3\sqrt{8+4*7}}} = \sqrt{6+2\sqrt{7+3\sqrt{8+4\sqrt{49}}}} \\ &= \cdots = \sqrt{6+2\sqrt{7+3\sqrt{8+4\sqrt{9+5\sqrt{10+\dots}}}}} \end{aligned}$$

It is plausible that having arrived at these exotic representations for 3 and 4, like a detective story writer, he posed the question, knowing its answer before hand! In his Notebooks he however noted down the more general result (p. 139, Entry 4 of the second Notebook of Ramanujan):

$$x+n+a=\sqrt{(ax+(n+a)^2+x\sqrt{(a(x+n)+(n+a)^2+(x+n)\sqrt{\text{etc.}}}}$$

from which by putting $x=2, n=1$ and $a=0$ or 1 one obtains the results (i) or (ii), respectively. Finally, it may be conjectured that this entry in his notebook was at about the time when his classmate and friend C.V. Rajagopalachari posed the question: if $\sqrt{x}+y=7$ and $x+\sqrt{y}=11$, what are the values of x and y? For which, Ramanujan shot back the answer $x=9$ and $y=4$.

- Q. 464: 2^n-7 is a perfect square for the values 3, 4, 5, 7, 15 of n. Find other values.

That the Diophantine equation

$$x^2+7=2^n$$

has only 5 solutions was proved by Nagell in 1940. A generalization of this equation:

$$x^2+D=p^n$$

is called the Ramanujan-Nagell equation. Prof. T.N. Shorey has shown that:

$$x^2+7=4y^n$$

has no solutions for $x\geq 1,\ y>2,\ n>1$.

- Q. 469: $1+n!$ is a perfect square for values 4, 5, 7 of n. Find other values.

The Diophantine equation:

$$1+n!=m^2$$

is called the Brocard-Ramanujan equation in literature today. Prof. H. Gupta (1935) has shown that there are no solutions to this equation for $8\leq n\leq 63$ and more recently (2000) Prof. Berndt and W. Galway showed that there are no solutions up to $n=10^9$. It is yet to be proved that the equation has only three solutions!

It will remain a mystery whether Ramanujan had proofs that there were no more solutions to the above two Diophantine equations (in Q. 464 and 469) and he still posed them as questions, in his inimitable style! In the case of the celebrated Rogers-Ramanujan identities, which were also posed by Ramanujan as Q. 584 in the JIMS, Ramanujan had no proof but when asked by Prof. Hardy he came up with a proof of the identities.

It is surprising that no serious effort has been made by the various curriculum development authorities for mathematics education in our country to include in the school/college syllabi some aspects of the mathematics of Ramanujan. However, at Panjab University, Prof. Ashok Agarwal has introduced, in 1998, an advanced Topics in Number Theory course for final year M.Sc. (Honours School) students. The syllabus includes basic hypergeometric series, Rogers-Ramanujan identities, Ramanujan's congruences and mock theta functions. This course has become popular with the students and augurs well for the future. It is our hope that Prof. R.P. Agarwal's *Resonance of Ramanujan's Mathematics* (3 Vols., New Age International (P) Ltd., 1996, 1999) would be also used as reference/text book material in such advanced courses in Number Theory. Perhaps in the new millennium, since Ramanujan's work, his conjectures, his questions in the JIMS and his tantalisingly recorded results in his Notebooks have been a source of inspiration and stimulated the research work of generations of mathematicians the world over, these would be included in syllabi. Prof. Atle Selberg, in 1988, at an extempore lecture at the Tata Institute of Fundamental Research observed that *a felicitous but unproved conjecture may be of much more consequence than the proof of many a respectable theorem.* The many conjectures of Ramanujan opened up new areas of research in mathematics.

A *πie* Pavilion – highlighting briefly the contributions of the Indian mathematicians, Euler, Gauss, Jacobi and the several formulae of Ramanujan including his 17 infinite series representations for π – and a replica of the Ramanujan Museum at Royapuram were designed and created for the Indian Science Congress Exhibition, at Chennai in Jan. 1999, by the author. Subsequently, steps were taken to house the exhibitions with some modifications in the Periyar Science and Technology Center (PSTC), which was opened for the public on Feb. 28, 1999. With additional funds from the Department of Science and Technology, Government of India, a *Ramanujan Photo Gallery* was added in June 1999, in the PSTC.

A Pilot CD-ROM Project on the Life and Work of Srinivasa Ramanujan created by the author for the Indian Science Congress Exhibition in 1999, held in Chennai, has been the precursor to a full length Project proposed by the Institute of Mathematical Sciences and the National Multimedia Resource Center of C-DAC, Pune. This has been approved by the Department of Science and Technology, Government of India, and the CD-ROM project is expected to be completed in 2002.

It is heartening to note that through a new initiative of Dr. Pon Kothandaraman, Vice Chancellor, University of Madras, the Ramanujan Institute and the Institute of Mathematical Sciences are planning an appropriate Museum to house the original Notebooks of Ramanujan and a few of the memorabilia connected with his life – such as his passport, the slate used by him, some of the original letters written by him or received by him. This live Museum is expected to stimulate the interest of generations of mathematicians, students and the public to the life and work of Ramanujan.

Freeman J. Dyson, the renowned Physicist concludes his article entitled: *A Walk through Ramanujan's Garden* – in *Ramanujan Revisited*, Proceedings of the Ramanujan Centenary conference, in 1987, at University of Illinois – with the following advise:

> *In conclusion, I would like to urge all of you who are working in the many fields of mathematics which have been enriched by Ramanujan's ideas to go back to the source the collected papers and the notebooks. ... The notebooks ... are now appearing in a splendidly annotated version edited by Bruce Berndt. The "lost" notebook is now accessible to us through the devoted labors of George Andrews. When I started my walk through Ramanujan's garden 47 years ago, only the collected papers were available. A year after I chose Hardy and Wright's "Theory of Numbers" (Oxford, Clarendon Press, 1938) as a school prize, I won another prize. For the second prize I chose Ramanujan's collected papers. The collected papers have traveled with me from England to America and are still as fresh to-day as they were in 1940. Whenever I am angry or depressed, I pull down the collected papers from the shelf and take a quiet stroll in Ramanujan's garden. I recommend this therapy to all of you who suffer from headaches or jangled nerves. And Ramanujan's papers are not only a good therapy for headaches. They also are full of beautiful ideas which may help you to do more interesting mathematics.*

Certainly several of us will not find this therapy quite as effective as it was for Dyson! Dyson, Erdös and Selberg were among the very best in their chosen fields and were greatly benefited by an early introduction to the *Collected Papers* of Ramanujan. However, it is not an exaggeration to say that the prolific creative work of the mathematical genius Ramanujan contained in his *Collected Papers*, in his original Notebooks and the 'Lost' Notebook, the study and lectures on Ramanujan's work by Prof. Hardy and the exhaustive editing of Ramanujan's Notebooks by Prof. Berndt, will be the sources of inspiration for many more generations of students of mathematics in the new millennium.

Acknowledgement

I wish to thank Prof. Ashok K. Agarwal for inviting me to write this article, in lieu of the actual contribution I made at the Chandigarh conference on my work on *Multiplicative Diophantine Equations*.

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Number of Solutions of Equations over Finite Fields and a Conjecture of Lang and Weil

Sudhir R. Ghorpade and Gilles Lachaud

A brief survey of the conjectures of Weil and some classical estimates for the number of points of varieties over finite fields is given. The case of partial flag manifolds is discussed in some details by way of an example. This is followed by a motivated account of some recent results on counting the number of points of varieties over finite fields, and a related conjecture of Lang and Weil. Explicit combinatorial formulae for the Betti numbers and the Euler characteristics of smooth complete intersections are also discussed.

Introduction

The general question that we want to discuss in this article may be stated in its simplest form as follows.

Suppose p is a prime number and $f(T_0, T_1, \ldots, T_N)$ is a polynomial with integer coefficients. Then what can we say about the solutions of the congruence equation $f(T_0, T_1, \ldots, T_N) \equiv 0 (\bmod p)$?

For example, if f is a quadratic polynomial in one variable of the form $T^2 - a$ (or alternatively, a homogeneous polynomial in two variables of the form $T_0^2 - T_1^2$), then the question is related to the classical study of quadratic residues. As in the classical case, by solutions we really mean solutions (mod p). In other words, we are interested in the set

$$V_f = \{(t_0, t_1, \ldots, t_N) \in \mathbb{F}_p^{N+1} : f(t_0, t_1, \ldots, t_N) = 0\},$$

where $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is the field with p elements. Note that since f has integer coefficients, the substitution of values from $\mathbb{F}_p$ in f makes sense. In case f is homogeneous, then $(t_0, t_1, \ldots, t_N)$ is a solution if and only if $(\lambda t_0, \lambda t_1, \ldots, \lambda t_N)$ is a solution for any $\lambda \in \mathbb{F}_p$, $\lambda \neq 0$, and the solutions that are proportional to

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each other ought to be considered equivalent. Thus it is better in this case to look at the set

$$X_f = \{(t_0 : t_1 : \cdots : t_N) \in \mathbb{P}^N_{\mathbb{F}_p} : f(t_0, t_1, \ldots, t_N) = 0\},$$

where $\mathbb{P}^N_{\mathbb{F}_p}$ denotes the N-dimensional projective space over $\mathbb{F}_p$.

Roughly speaking, sets such as V_f and X_f are known as affine varieties and projective varieties, respectively, especially when f is irreducible. Better still, they are the sets of $\mathbb{F}_p$-rational points of affine or projective varieties, and to emphasize this, the notation $V_f(\mathbb{F}_p)$ or $X_f(\mathbb{F}_p)$ is sometimes used. An important feature of these is that they are finite sets. In fact, instead of $\mathbb{F}_p$, we could look more generally at any finite field $\mathbb{F}_q$ with q elements and the corresponding sets $V_f(\mathbb{F}_q)$ or $X_f(\mathbb{F}_q)$ are well defined and finite. So the general problem mentioned above can, in part, be stated as follows.

Given an affine or projective variety X defined over the finite field $\mathbb{F}_q$, count or estimate the cardinality $|X(\mathbb{F}_q)|$ of the set of $\mathbb{F}_q$-rational points of X.

In this connection, André Weil made a remarkable observation about fifty years ago. Namely, that the arithmetic or combinatorial question of counting the number of points of varieties over finite fields is intimately related to the topology of related objects such as the corresponding variety over the complex numbers. His observation was formulated in the form of a number of conjectures, which are now theorems. This work leads to some nontrivial estimates for the number $|X(\mathbb{F}_q)|$. In the special case such as X_f when the variety is a *hypersurface*, that is, it is defined by a single equation, or in the more general case when it is a *complete intersection*, the estimates one gets are particularly sharp provided that the variety is nonsingular.

In the next section, we shall give a brief introduction to the celebrated conjectures of Weil and some related results, with an emphasis on the resulting estimates for the number of $\mathbb{F}_q$-rational points of varieties over finite fields. The example of partial flag manifolds is worked out in a fairly self-contained manner in the subsequent section, so as to provide an illustration of Weil Conjectures, and also to serve as a ready reference for some material that may be of independent interest. We then turn to certain recent results, where some of the classical estimates for the number of $\mathbb{F}_q$-rational points of varieties over finite fields are extended or refined, and a related conjecture of Lang and Weil is proven. Our aim here is to provide a short and leisurely introduction to these recent results and some background material, in such a way that it can be easily accessible to the non-experts. A reader who wishes to know more may consult [8], [28] and [10] where further details can be found. Additional pointers to the literature appear in the subsequent sections.

1. Weil Conjectures and Classical Estimates

From this section onwards, it will be convenient to use some language of Algebraic Geometry. For most part, it will suffice to know what an affine or projective variety is, and some basic things such as the dimension of a variety, the notion of nonsingular or smooth varieties, etc. In essence, an affine variety over a field k is the locus in the affine space $\mathbb{A}_k^N$ (which means the space of N-tuples of elements from k) of a bunch of polynomial equations in N variables with coefficients in k, while a projective variety is the locus in the projective space $\mathbb{P}_k^N$ of a bunch of homogeneous polynomials in $N + 1$ variables with coefficients in k. For more details concerning the basics of Algebraic Geometry, one may consult standard texts such as Hartshorne [13] or Shafarevich [37].

Let X be a variety defined over the finite field $\mathbb{F}_q$. Then X is also defined over any extension $\mathbb{F}_{q^r}$ of $\mathbb{F}_q$. The numbers $a_r = |X(\mathbb{F}_{q^r})|$ of $\mathbb{F}_{q^r}$-rational points of X are therefore well-defined. Now, the exponential generating function of these numbers, namely,

$$Z(X,T) = \exp\left(\sum_{r=1}^{\infty} a_r \frac{T^r}{r}\right)$$

is a well-defined formal power series with rational coefficients. This is called the *(Weil) zeta function* of X.

For example, if $X = \mathbb{P}^n$, the projective space of dimension n over $\mathbb{F}_q$, then we can easily see that

$$|\mathbb{P}^n(\mathbb{F}_q)| = \pi_n := q^n + q^{n-1} + \cdots + q + 1.$$

Of course, a similar formula holds with q replaced by q^r. Thus,

$$Z(X,T) = \exp\left(\sum_{r=1}^{\infty} \frac{T^r}{r} \sum_{j=0}^{n} q^{rj}\right) = \prod_{j=0}^{n} \frac{1}{1 - q^j T},$$

where the last equality follows by remembering the familiar expansion of $\log(1 - q^j T)$. It may be noted here that $Z(X,T)$ is, in fact, a rational function in T, and it satisfies the equation $Z(X, 1/(q^n T)) = (-q^{n/2}T)^{n+1} Z(X,T)$. The conjectures of Weil state that the situation in the general case is similar. More precisely, these conjectures may be stated as follows. Let X be a nonsingular projective variety of dimension n. Then we have the following.

(i) $Z(X,T)$ is a rational function in T.

(ii) $Z(X,T)$ satisfies the functional equation

$$Z\left(X, \frac{1}{q^n T}\right) = (-q^{n/2}T)^{\chi} Z(X,T), \tag{1}$$

for some integer χ. More precisely, there is a factorization

$$Z(X,T) = \frac{P_1(X,T)P_3(X,T)\cdots P_{2n-1}(X,T)}{P_0(X,T)P_2(X,T)\cdots P_{2n}(X,T)},$$

where $P_0(X,T) = 1 - T$, $P_{2n}(X,T) = 1 - q^n T$, and for $0 \le i \le 2n$, $P_i(X,T)$ are polynomials with integer coefficients such that if $b_i = \deg P_i(X,T)$, then

$$T^{b_i} P_i\left(X, \frac{1}{q^n T}\right) = (-1)^{b_i} q^{(i-n)b_i/2} P_{2n-i}(X,T).$$

(This implies that $b_i = b_{2n-i}$ and (1) holds with $\chi = \sum_{i=0}^{2n} (-1)^i b_i$.)

(iii) If $P_i(X,T)$ are as above, then there is a factorization

$$P_i(X,T) = \prod_{j=1}^{b_i} (1 - \omega_{ij} T), \tag{2}$$

where the reciprocal roots ω_{ij} are algebraic integers with the property that $|\omega_{ij}| = q^{i/2}$ for $1 \le j \le b_i$ and $0 \le i \le 2n$.

(iv) If X is obtained from 'reduction modulo p' of a nonsingular projective variety Y over $\mathbb{Z}$ or more generally, over a ring of algebraic integers, and if $Y_{\mathbb{C}}$ is the complex manifold corresponding to Y, then $b_i = \deg P_i(X,T)$ equals the ith Betti number of $Y_{\mathbb{C}}$. In particular, the number χ in (ii) is the Euler characteristic of $Y_{\mathbb{C}}$.

These conjectures were proved in the case of curves, i.e., when $n = 1$, by Weil himself in the 40's. In this particular case when X is a nonsingular projective curve, we have $b_0 = b_2 = 1$ and $b_1 = 2g$ where g is the *genus* of X. Thus if we write ω_j for ω_{1j}, then from (ii) and (iii), we see that

$$a_r = |X(\mathbb{F}_{q^r})| = 1 + q^r - \sum_{j=1}^{2g} w_j^r. \tag{3}$$

Now $|\omega_j| = \sqrt{q}$, and so we obtain our first classical estimate:

$$||X(\mathbb{F}_q)| - \pi_1| \le 2g\sqrt{q} \quad \text{where } \dim X = 1 \text{ and } \pi_1 = 1 + q. \tag{4}$$

This is known as *Weil's inequality*. We remark that if in (3), we write $\omega_j = \sqrt{q}e^{2\pi i\theta_j}$, where $0 \le \theta_j \le 1$, then by a classical theorem of Kronecker (see, for example, [12, Ch. XXIII]), for every $\epsilon > 0$, we can find (infinitely many) $r \ge 1$ such that

$$\left| \sum_{j=1}^{2g} e^{2\pi i r\theta_j} - 2g \right| < \epsilon.$$

Thus it follows that the constant $2g$ in (4) is the best possible, in general. However, for a given q, improvements can be made. This was observed, for example, by Serre who refined (4) to obtain the following estimate.

$$||X(\mathbb{F}_q)| - \pi_1| \le g[2\sqrt{q}] \quad \text{where } \dim X = 1 \text{ and } \pi_1 = 1 + q. \tag{5}$$

Here $[2\sqrt{q}]$ denotes, as usual, the greatest integer $\leq 2\sqrt{q}$. For example, if $q = 2$ and $g = 50$, the Weil bound for $|X(\mathbb{F}_q)|$ is 144 while the Serre bound is 103.

The genus g of a nonsingular projective curve X is, in general, difficult to determine, especially if we describe X by its defining equations. However, if we fix an embedding of X in a projective space, and let d denote the degree of X, then the genus satisfies the bound $g \leq (d-1)(d-2)/2$. Thus (4) implies a similar inequality with $2g$ replaced by $(d-1)(d-2)$. This weaker inequality was generalized by Lang and Weil [24] in 1954 to varieties of arbitrary dimension. They proved that if X is a projective variety in $\mathbb{P}^N$ defined over $\mathbb{F}_q$ and of dimension n and degree d, then

$$(6) \qquad ||X(\mathbb{F}_q)| - \pi_n| \leq (d-1)(d-2)q^{n-(1/2)} + Cq^{n-1},$$

where C is a constant depending only on N, n and d (so that it is independent of q). We will refer to (6) as the *Lang-Weil inequality*.

The first part of Weil's conjecture about the rationality of $Z(X, T)$ was proved in 1960 by Dwork [7]. In fact, the rationality was conjectured and proved for arbitrary varieties that are not necessarily projective or nonsingular. Dwork's proof used methods from p-adic analysis, which were quite different from the heuristic approach suggested by Weil for proving his conjectures. Weil's idea was based on the simple observation that an element a in the algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$ is in $\mathbb{F}_q$ if and only if $a^q = a$. In other words, a is the fixed point of the Frobenius map $F : \overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q$ which sends x to x^q. The Frobenius map can be extended to an affine or projective space over $\overline{\mathbb{F}}_q$ by simply raising each coordinate to its qth power. It follows, then, that the number of $\mathbb{F}_q$-rational points of a variety X (in the projective space over $\overline{\mathbb{F}}_q$) is the number of fixed points of X under the Frobenius map. Similarly, the points of $X(\mathbb{F}_{q^r})$ correspond to the fixed points of the r-fold composite F^r of the Frobenius map. Now we remember from Topology, the Lefschetz Fixed Point Formula which says that if Y is a complex manifold and $f : Y \to Y$ is a map with isolated fixed points, then the number of fixed points of f^r is the alternating sum of the traces of the (linear) maps induced by f^r on the cohomology spaces of Y. Thus,

$$\text{Number of fixed points of } f^r = \sum_{i=0}^{2n} (-1)^i \operatorname{Tr}\,(f^r | H^i(Y, \mathbb{C})).$$

To carry this idea to varieties over finite fields, it was felt desirable to develop a suitable cohomology theory in which a similar trace formula holds. This task was accomplished by Artin and Grothendieck around 1963. They showed that if p is the characteristic of $\mathbb{F}_q$ and ℓ is any prime different from p, then to a projective variety X over $\mathbb{F}_q$, we can associate the *étale ℓ-adic cohomology spaces* $H^i(\bar{X}, \mathbb{Q}_\ell)$. Here, $\bar{X} = X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ denotes the variety corresponding to X over the algebraic closure of $\mathbb{F}_q$, and $\mathbb{Q}_\ell$ denotes, as usual, the field of ℓ-adic numbers. These cohomology spaces are finite-dimensional vectors spaces over $\mathbb{Q}_\ell$. Moreover, if $n = \dim X$, then $H^i(\bar{X}, \mathbb{Q}_\ell)$ vanish if $i < 0$ or $i > 2n$. The Frobenius map on $\bar{X}$ induces an endomorphism of $H^i(\bar{X}, \mathbb{Q}_\ell)$, and the following analogue of the Lefschetz Trace Formula holds.

$$|X(\mathbb{F}_{q^r})| = \sum_{i=0}^{2n} (-1)^i \operatorname{Tr} (F^r | H^i(\bar{X}, \mathbb{Q}_\ell)). \tag{7}$$

We will refer to (7) as the *Grothendieck-Lefschetz Trace Formula*. Now if we let $P_{i,\ell}(X, T) = \det(1 - TF|H^i(\bar{X}, \mathbb{Q}_\ell))$, for $0 \le i \le 2n$, then it is a formal and easy consequence of (7) that

$$Z(X, T) = \frac{P_{1,\ell}(X, T) P_{3,\ell}(X, T) \cdots P_{2n-1,\ell}(X, T)}{P_{0,\ell}(X, T) P_{2,\ell}(X, T) \cdots P_{2n,\ell}(X, T)} . \tag{8}$$

Thus, the rationality of $Z(X, T)$ is proved. Notice that if we factor $P_{i,\ell}(X, T)$ as in (2), then we see that the reciprocal roots $\omega_{ij} = \omega_{ij,\ell}$ are the eigenvalues of the Frobenius endomorphism on $H^i(\bar{X}, \mathbb{Q}_\ell)$, and so we can rewrite (7) as

$$|X(\mathbb{F}_{q^r})| = \sum_{i=1}^{2n} (-1)^i \sum_{j=1}^{b_{i,\ell}} \omega_{ij}^r, \quad \text{where } b_{i,\ell} = \dim_{\mathbb{Q}_\ell} H^i(\bar{X}, \mathbb{Q}_\ell). \tag{9}$$

In case X is nonsingular, then it was shown that the ℓ-adic Betti numbers $b_{i,\ell}$ are independent of ℓ and moreover, Poincaré duality holds, which means, in particular, that $b_{i,\ell} = b_{2n-i,\ell}$ for $0 \le i \le 2n$. These facts and some elementary Linear Algebra imply the functional equation for $Z(X, T)$ conjectured by Weil. Independent proofs of rationality and the functional equation for $Z(X, T)$ were also given by Lubkin [25, 26] around 1967.

For the étale cohomology spaces $H^i(\bar{X}, \mathbb{Q}_\ell)$, a *Comparison Theorem* holds in the case when X is nonsingular and obtained by 'reduction mod p' from a nonsingular variety in characteristic zero. This implies (iv) in the list of Weil conjectures. Thus, what remains to be proved is (iii), which is commonly referred to as the *Riemann hypothesis*[1] for varieties over finite fields. This Riemann hypothesis was finally proved by Deligne [5] in 1973. As a first application of his work, Deligne [5, Thm. 8.1] gave the following estimate for the number of points of nonsingular complete intersections:

$$||X(\mathbb{F}_q)| - \pi_n| \le b'_n q^{n/2}. \tag{10}$$

Here X is assumed to be a nonsingular complete intersection in $\mathbb{P}^N$, of dimension n, defined over $\mathbb{F}_q$ and b'_n is its primitive nth Betti number. We will explain these terms and describe explicit formulae for b'_n in Section 3. Returning to the general case, another proof of the Riemann hypothesis and a weaker version of it for varieties that are not necessarily nonsingular, was obtained by Deligne in a subsequent paper [6]. In effect, Deligne's result for arbitrary projective varieties states that

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[1] This nomenclature will be clear if we translate (iii) as a condition for the function $\zeta(X, s) = Z(X, q^{-s})$.

(11) *the eigenvalues* $\omega_{ij,\ell}$ *of* $F \mid H_c^i(\bar{X}, \mathbb{Q}_\ell)$ *are pure of weight* $\leq i$.

Here, we call $\alpha \in \bar{\mathbb{Q}}_\ell$ *pure of weight* r if α is an algebraic integer and if $|\iota(\alpha)| = q^{r/2}$ for any embedding ι of $\bar{\mathbb{Q}}_\ell$ into $\mathbb{C}$. We will refer to (11) as *Deligne's Main Theorem.*

Both Grothendieck-Lefschetz Trace Formula (7) and Deligne's Main Theorem (11) are, in fact, valid for objects X that are more general than projective varieties (namely, separated schemes of finite type[2] over $\mathbb{F}_q$, of dimension n), and these two results are the most fundamental tools for much of the subsequent work on the number of points of varieties over finite fields.

To end this section, we provide some pointers to the references for the results described in this section. Weil's conjectures were formally stated in his article [42] in the *Bulletin*. The improvement in Weil's inequality by Serre is given in [36]. The proof of Serre bound is rather short and elementary — it mainly uses things like the AM-GM inequality! For the Lang-Weil inequality, the original paper [24] is probably the best reference. An overview of Dwork's proof [7] can be found in the memorial article by Katz and Tate [21] in the *Notices*. The idea of using a suitable analogue of the Lefschetz fixed point formula for counting points of varieties over finite fields appears in Weil's ICM address [43]. The theory of étale topology first appeared in the Harvard notes of Artin [2], and later in SGA [3]. One of the earlier elementary exposition can be found in an article [32] by Mumford. A more recent review is given by Katz [18]. There is also the book of Milne [28] as well as his lecture notes [29] that are more elementary and available for free! An overview of Deligne's proof of Riemann hypothesis and some of its spectacular applications such as the proof of Ramanujan-Peterson conjecture, is given in the article of Katz [17]. The book by Freitag and Kiehl [8] contains a proof of Weil conjectures along with some background material and furthermore, an article by Dieudonné giving a historical introduction to Weil conjectures. An overview of the Weil conjectures also appears in an appendix to Hartshorne's book [13]. There are now shorter proofs of the Riemann hypothesis for varieties over finite fields: by Laumon in 1984 and most recently by Katz [20]. For an account of Laumon's work and for more on Deligne's Main Theorem (11), we refer to the recent book [22] by Kiehl and Weissauer.

2. An Example

In this section, we shall illustrate the conjectures of Weil and the Trace Formula by a rather substantial example. This example involves the partial flag manifolds, which are not only of interest in Algebraic Geometry, but also in Combinatorics, Representation Theory and Topology. However, for the general results described in later sections, this example is of no direct relevance, and it may be skipped at a first reading.

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[2] In the case of schemes of this type, it is better to work with étale ℓ-adic cohomology with *compact supports*.

Let k be a field and V be a vector space of dimension m over k. Let $\underline{\ell} = (\ell_1, \dots, \ell_s)$ be a sequence of integers such that $0 < \ell_1 < \cdots < \ell_s < m$. By a *partial flag* of dimension $\underline{\ell}$ we mean a sequence $(V_1, \dots, V_s)$ of subspaces of V such that $V_1 \subset \cdots \subset V_s$ and $\dim V_i = \ell_i$ for $1 \le i \le s$. Let $\mathcal{F}_{\underline{\ell}}(V)$ denote the set of partial flags of dimension $\underline{\ell}$. Two special cases of this arise more frequently. If $s = 1$ and we write $l = \ell_1$, then $\mathcal{F}_{\underline{\ell}}(V)$ is the *Grassmannian* $\mathrm{Gr}_l(V)$ of l-planes in the m-space (over k). At the other extreme, if $s = m - 1$, then we necessarily have $\ell_i = i$ for $1 \le i \le s$, and in this case $\mathcal{F}_{\underline{\ell}}(V)$ is the space of *full flags* or *complete flags*. It is elementary and well-known that any Grassmannian embeds in a projective space[3] as a projective variety given by the vanishing of certain quadratic homogeneous polynomials. Using this, we obtain natural maps

$$\mathcal{F}_{\underline{\ell}}(V) \hookrightarrow \mathrm{Gr}_{\ell_1}(V) \times \cdots \times \mathrm{Gr}_{\ell_1}(V) \hookrightarrow \mathbb{P}\left(\bigwedge^{\ell_1} V\right) \times \cdots \times \mathbb{P}\left(\bigwedge^{\ell_s} V\right)$$

and the latter product of projective spaces embeds naturally into a big projective space via the so called Segre embedding. In this way, $\mathcal{F}_{\underline{\ell}}(V)$ becomes a projective algebraic variety, and the defining equations are known to have integer coefficients (see, for example, [9, Sec. 9.1] or [16, Sec. 1.8]). Moreover, $\mathcal{F}_{\underline{\ell}}(V)$ is smooth and to see this, it suffices to note that $\mathcal{F}_{\underline{\ell}}(V)$ is a homogeneous space of the form G/P, where $G = \mathrm{GL}(V)$ is the group of invertible linear transformations of V (which is identified with $\mathrm{GL}_m(k)$ once we fix a basis of V) and P is the stabilizer of a partial flag with respect to the natural action of G on $\mathcal{F}_{\underline{\ell}}(V)$[4].

Now suppose $k = \mathbb{F}_q$. Then we can easily calculate the number $|\mathcal{F}_{\underline{\ell}}(V)(\mathbb{F}_q)|$ of $\mathbb{F}_q$-rational points of $\mathcal{F}_{\underline{\ell}}(V)$. For this, first note that the number of l-dimensional subspaces W of the m-dimensional vector space V over $\mathbb{F}_q$ can be computed as follows. A basis $\{w_1, \dots, w_l\}$ of a l-dimensional subspace is obtained by successively choosing w_1 in $V \setminus \{0\}$, w_2 in $V \setminus (\mathbb{F}_q w_1)$, w_3 in $V \setminus (\mathbb{F}_q w_1 + \mathbb{F}_q w_2)$, and so on. Thus, there are $(q^m - 1)(q^m - q)(q^m - q^2) \cdots (q^m - q^{l-1})$ choices. Two bases determine the same subspace if and only if the corresponding $l \times m$ matrices differ by a factor in $\mathrm{GL}_l(\mathbb{F}_q)$. The cardinality of $\mathrm{GL}_l(\mathbb{F}_q)$ is the number of choices of l linearly independent row vectors in $\mathbb{F}_q^m$, and arguing as before, we see that this number is $(q^l - 1)(q^l - q) \cdots (q^l - q^{l-1})$. It follows that

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[3] This embedding is known as *Plücker embedding*, and it may be described as follows. To a subspace W in the Grassmannian $\mathrm{Gr}_l(V)$, we associate the point of the projective space $\mathbb{P}(\bigwedge^l V) = \mathbb{P}_k^{\binom{m}{l}-1}$ corresponding to the 1-dimensional subspace of the exterior product $\bigwedge^l V$ spanned by $w_1 \wedge \cdots \wedge w_l$, where $\{w_1, \dots, w_l\}$ is any k-basis of W.

[4] Fix a basis $\{e_1, \dots, e_m\}$ of V. If we consider the partial flag $(E_{\ell_1}, \dots, E_{\ell_s})$ where $E_{\ell_i} = \mathrm{span}\{e_1, \dots, e_{\ell_i}\}$, then its stabilizer P corresponds to matrices $g = (g_{uv}) \in \mathrm{GL}_m(k)$ with $g_{uv} = 0$ for $\ell_{i-1} < u \le \ell_i$ and $1 \le v \le \ell_i$ $(i = 1, \dots, s + 1)$ with the convention that $\ell_0 = 0$ and $\ell_{s+1} = m$.

$$|\mathrm{Gr}_l(V)(\mathbb{F}_q)| = \begin{bmatrix} m \\ l \end{bmatrix}_q := \frac{(q^m - 1)(q^m - q) \cdots (q^m - q^{l-1})}{(q^l - 1)(q^l - q) \cdots (q^l - q^{l-1})}.$$

The expression on the right is known as the *Gaussian binomial coefficient* corresponding to m and l. As for calculating $\mathcal{F}_{\underline{\ell}}(V)(\mathbb{F}_q)$, we note that the inclusion relations in a partial flag $(V_1, \ldots, V_s)$ will disappear if we consider instead the subspaces $V_1, V_2/V_1, \ldots, V_s/V_{s-1}$ of $V, V/V_1, \ldots, V/V_{s-1}$, respectively. This leads to the following formula for the number of $\mathbb{F}_q$-rational points of $\mathcal{F}_{\underline{\ell}}(V)$ as a *Gaussian multinomial coefficient*:

$$|\mathcal{F}_{\underline{\ell}}(V)(\mathbb{F}_q)| = \begin{bmatrix} m \\ \ell_1, \ell_2 - \ell_1, \ldots, \ell_s - \ell_{s-1}, m - \ell_s \end{bmatrix}_q := \prod_{i=1}^{s+1} \begin{bmatrix} m - \ell_{i-1} \\ \ell_i - \ell_{i-1} \end{bmatrix}_q$$

with the convention that $\ell_0 = 0$ and $\ell_{s+1} = m$.

Now suppose $k = \mathbb{C}$. Then $\mathcal{F}_{\underline{\ell}}(V)$ is a compact complex manifold[5]. Moreover, its Betti numbers can be computed easily since we have a nice cellular decomposition. This decomposition is obtained as follows. Fix a basis $\{e_1, \ldots, e_m\}$ of V, and consider the standard full flag $(E_1, \ldots, E_{m-1})$ where $E_j = \operatorname{span}\{e_1, \ldots, e_j\}$ for $1 \leq j \leq m-1$. Set $E_0 = \{0\}$ and $E_m = V$. Given any $(V_1, \ldots, V_s) \in \mathcal{F}_{\underline{\ell}}(V)$, set $V_{s+1} = V$, $\ell_{s+1} = m$, and define $r(i,j) = \dim V_i \cap E_j$ for $1 \leq i \leq s+1$ and $0 \leq j \leq m$. Note that

$$0 = r(i,0) \leq r(i,1) \leq r(i,2) \leq \cdots \leq r(i,m) = \ell_i \quad \text{for } i = 1, \ldots, s+1.$$

The consecutive terms of the sequence $(r(i,1), \ldots, r(i,m))$ differ by at most 1 [because the jth coordinate map $V_i \cap E_j \to \mathbb{C}$ has kernel $V_i \cap E_{j-1}$], and there are exactly ℓ_i jumps in this sequence [a *jump* is an index $j \in \{1, \ldots, m\}$ such that $r(i,j) = r(i,j-1) + 1$]. Moreover, since $V_{i+1} \cap E_{j-1} = V_{i+1} \cap E_j$ implies that $V_i \cap E_{j-1} = V_i \cap E_j$, we see that the ℓ_i jumps in $(r(i,1), \ldots, r(i,m))$ are included in the ℓ_{i+1} jumps in $(r(i+1,1), \ldots, r(i+1,m))$. It follows that there is a unique permutation σ of $\{1, \ldots, m\}$ such that $\sigma(1), \ldots, \sigma(\ell_i)$ are precisely the jumps in $(r(i,1), \ldots, r(i,m))$, and

$$(12) \qquad \sigma(\ell_{i-1}+1) < \sigma(\ell_{i-1}+2) < \cdots < \sigma(\ell_i) \quad \text{for } i = 1, \ldots, s+1.$$

We let $W_{\underline{\ell}}$ denote the set of permutations $\sigma \in S_m$ satisfying (12). The dimension array $(r(i,j))$ is determined by the corresponding $\sigma \in W_{\underline{\ell}}$ as follows:

$$r(i,j) = r_\sigma(i,j), \text{ where } r_\sigma(i,j) := |\{u \in \{1, \ldots, \ell_i\} : \sigma(u) \leq j\}|.$$

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[5] When $k = \mathbb{C}$, we can consider orthonormal bases, and note that there is a transitive action of the unitary group $U(n)$ on the partial flags so that $\mathcal{F}_{\underline{\ell}}(V)$ is a quotient of $U(n)$. The compactness of $\mathcal{F}_{\underline{\ell}}(V)$ is essentially a consequence of the fact that $U(n)$ is a compact group.

Therefore, we have a decomposition of $\mathcal{F}_{\underline{\ell}}(V)$ as a disjoint union:

$$\mathcal{F}_{\underline{\ell}}(V) = \coprod_{\sigma \in W_{\underline{\ell}}} \Omega_\sigma, \tag{13}$$

where for any $\sigma \in W_{\underline{\ell}}$,

$$\Omega_\sigma = \left\{ (V_1, \dots, V_s) \in \mathcal{F}_{\underline{\ell}}(V) : \begin{array}{c} \dim V_i \cap E_j = r_\sigma(i,j), \text{ for} \\ 1 \le i \le s+1, \ 1 \le j \le m \end{array} \right\}.$$

We now show that every Ω_σ is isomorphic to a cell, i.e., an affine space over $\mathbb{C}$. Fix $\sigma \in W_{\underline{\ell}}$. Given $(V_1, \dots, V_s) \in \Omega_\sigma$, we can recursively find vectors $\mathbf{x}_1, \dots, \mathbf{x}_m$ in V such that $\{\mathbf{x}_1, \dots, \mathbf{x}_{\ell_i}\}$ is a basis of $V_i = V_i \cap E_m$, and more generally, for $1 \le i \le s+1$ and $1 \le j \le m$,

$$\{\mathbf{x}_u : 1 \le u \le \ell_i \text{ and } \sigma(u) \le j\} \text{ is a basis of } V_i \cap E_j, \tag{14}$$

and further, if $(x_{u1}, \dots, x_{um})$ are the coordinates of $\mathbf{x}_u$ (with respect to the basis $\{e_1 \dots, e_m\}$ of V), then

$$x_{u\sigma(u)} = 1, \ x_{uv} = 0 = x_{u\sigma(u')} \text{ for } v > \sigma(u) \text{ and } u' < u, \tag{15}$$

where u, v, u' vary in $\{1, \dots, m\}$. The partial flag $(V_1, \dots, V_s) \in \Omega_\sigma$ and the matrix (x_{uv}) with rows $\mathbf{x}_1, \dots, \mathbf{x}_m$ satisfying (14) and (15) clearly determine each other uniquely. Moreover from (15) it is clear that the number of 'free entries' in this matrix are those x_{uv} for which $v = \sigma(u')$ with $u < u'$ and $\sigma(u) > \sigma(u')$. It follows that

$$\Omega_\sigma \simeq \mathbb{C}^{\mathrm{inv}(\sigma)} \simeq \mathbb{R}^{2\mathrm{inv}(\sigma)},$$

where $\mathrm{inv}(\sigma)$ denotes the number of inversions in the permutation σ. Therefore, (13) is a cellular decomposition of $\mathcal{F}_{\underline{\ell}}(V)$. Since there are no (real) cells in odd dimensions, it follows from standard topological arguments that the odd dimensional Betti numbers of $\mathcal{F}_{\underline{\ell}}(V)$ are zero while

$$b_{2\nu}(\mathcal{F}_{\underline{\ell}}(V)) = |\{\sigma \in W_{\underline{\ell}} : \mathrm{inv}(\sigma) = \nu\}|, \quad \text{for } \nu \ge 0. \tag{16}$$

The reason why the formula for the number of $\mathbb{F}_q$-rational points of $\mathcal{F}_{\underline{\ell}}(V)$ is in accordance with (16) and the Weil conjectures is the following combinatorial identity that is about 90 years old and goes back to MacMahon (cf [1, Thm. 3.4]):

$$\begin{bmatrix} m \\ \ell_1, \ell_2 - \ell_1, \dots, \ell_s - \ell_{s-1}, m - \ell_s \end{bmatrix}_q = \sum_{\nu \ge 0} \mathrm{Inv}(\underline{\ell}; \nu) q^\nu, \tag{17}$$

where $\mathrm{Inv}(\underline{\ell}; \nu)$ denotes the number of permutations $\tau_1 \tau_2 \dots \tau_m$ of the multiset $M_{\underline{\ell}} := \{1^{\ell_1} 2^{\ell_2 - \ell_1} \dots s^{\ell_s - \ell_{s-1}} (s+1)^{m-\ell_s}\}$ with exactly ν inversions[6]. There is a natural inversion-preserving bijection between the permutations in $W_{\underline{\ell}}$ and the permutations of the multiset $M_{\underline{\ell}}$[7]. Thus, from (16) and (17), we obtain the identity

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[6] Just like permutations, the inversions of a multiset permutation $\tau_1 \tau_2 \dots \tau_m$ are the pairs (τ_i, τ_j) such that $1 \le i < j \le m$ and $\tau_i > \tau_j$.

$$|\mathcal{F}_{\underline{\ell}}(V)(\mathbb{F}_q)| = \sum_{\nu \geq 0} b_{2\nu}(\mathcal{F}_{\underline{\ell}}(V))q^{\nu},$$

which implies that the zeta function of $\mathcal{F}_{\underline{\ell}}(V)(\mathbb{F}_q)$ is a rational function of the desired form and the Riemann hypothesis is satisfied.

Remarks 2.1. 1. The cellular decomposition (13) and the equivalence between Ω_σ and an affine space of dimension $\operatorname{inv}(\sigma)$ is, in fact, valid when $\mathbb{C}$ is replaced by any field k. If we take $k = \mathbb{F}_q$, then in view of the above-mentioned bijection between $W_{\underline{\ell}}$ and multiset permutations, we obtain another proof of MacMahon's identity (17).

2. In another direction, we can use MacMahon's identity (17) together with Weil Conjectures to deduce the description (16) for the Betti numbers, and consequently, a description of the Poincaré polynomial of $\mathcal{F}_{\underline{\ell}}(V)$ as the multinomial coefficient in (17) with q replaced by q^2. Here, one has to make use of the fact that $\mathcal{F}_{\underline{\ell}}(V)$ is smooth and the number of $\mathbb{F}_{q^r}$-rational points of $\mathcal{F}_{\underline{\ell}}(V)$ is a polynomial in q^r, for every $r \geq 1$. Alternative descriptions of $b_{2\nu}(\mathcal{F}_{\underline{\ell}}(V))$ are also possible using combinatorial identities such as MacMahon's identity for $\operatorname{Inv}(\underline{\ell}\,; \nu)$ in terms of the number of multiset permutations with 'major index' ν. In the Grassmannian case ($s = 1$), this Betti number has an interpretation in terms of partitions, namely $b_{2\nu}(\mathrm{Gr}_l(V))$ is the number of partitions of ν with at most l parts, each part $\leq m - l$. For proofs of these combinatorial facts, we refer to [1, Sec. 3.4].

3. Using the cellular decomposition (13), or alternatively, the formula for the number of $\mathbb{F}_q$-rational points of $\mathcal{F}_{\underline{\ell}}(V)$ coupled with Weil conjectures, we see that the dimension of $\mathcal{F}_{\underline{\ell}}(V)$ (as a projective variety) is equal to

$$\sum_{1 \leq i < j \leq s+1} (\ell_i - \ell_{i-1})(\ell_j - \ell_{j-1})$$
$$= \binom{m+1}{2} - \sum_{i=1}^{s+1} \binom{\ell_i - \ell_{i-1} + 1}{2}.$$

This number is simply $l(m-l)$ in the Grassmannian case ($s = 1$ and $l = \ell_1$), whereas in the case of the full flag manifold ($s = m - 1$), this is $m(m-1)/2$.

4. We indicated earlier that $\mathcal{F}_{\underline{\ell}}(V)$ can be identified as a quotient G/P of $G = \mathrm{GL}(V)$. In this setting, $W_{\underline{\ell}}$ can be identified as W/W_P, where W and

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[7] This bijection is obtained as follows. Given $\sigma \in W_{\underline{\ell}}$, let τ be the permutation of $M_{\underline{\ell}}$ with i in the spots marked by $\sigma(\ell_{i-1} + 1), \ldots, \sigma(\ell_i)$, that is, $\tau_j = i$ if $\ell_{i-1} < \sigma^{-1}(j) \leq \ell_i$. For example, if $\ell_1 = 4$, $\ell_2 = 6$, $\ell_3 = 7$ and $\ell_4 = 8 = m$, then the multiset permutation corresponding to $\sigma = 12586734$ is $\tau = 11341221$.

W_P are the Weyl groups of G and P, respectively [in fact, $W \simeq S_m$ whereas $W_P \simeq S_{\ell_1} \times S_{\ell_2 - \ell_1} \times \cdots \times S_{\ell_{s+1} - \ell_s}$]. Moreover, if we let B denote the subgroup of G consisting of all upper triangular invertible matrices, then the cell Ω_σ can be identified with the double coset $B\sigma P/P$. Thus (13) is an instance of the so called Bruhat decomposition. For more on this point of view, see, for example, [16].

Basic facts about Grassmannians can be found in several texts. The cellular decomposition for Grassmannians is explained beautifully in the book of Milnor and Stasheff [31]. The relation of the Betti numbers of Grassmannians with the Gaussian binomial coefficient, in light of Weil conjectures, is pointed out, for instance, in Stanley's paper [41]. This paper also discusses the example of partial flag manifolds, albeit rather briefly. As remarked by Stanley [41], the known results about Grassmannians were one of the original pieces of evidence for the Weil conjectures. The cellular decomposition for full flag manifolds is given nicely in the *cours* of Manivel [27, Sec. 3.6]. A relatively recent book by Fulton [9] contains a good exposition of the basic facts about partial flag manifolds and the connection with Representation Theory. Although the cellular decomposition and a presentation for the cohomology (á la Borel) is discussed mostly in the case of full flag manifolds, a quick outline of its generalizations to partial flag manifolds is given towards the end of Ch. 10 in [9]. As for the various combinatorial facts mentioned in this section, it suffices to consult the encyclopedia volume of Andrews [1]; see also the book of Stanley [40] for some of this material and more.

3. Complete Intersections over Finite Fields

An algebraic variety defined by a single equation is called a hypersurface. This has codimension 1 in the ambient space. Conversely, if a variety is of codimension 1, then it can be shown that the variety can be defined by a single equation (see, for example, [37, Ch. 1, §6]). For arbitrary varieties, a similar result is not true. In case a variety of codimension r is defined by r equations, it is called a complete intersection. To be more precise, by a *complete intersection* over a field k, we shall mean in this article an irreducible projective variety[8] X in $\mathbb{P}_k^N$ of dimension n such that its homogeneous ideal $I(X)$ can be generated by $N - n$ homogeneous polynomials in $N + 1$ variables over k. If $f_1, \dots, f_r$ is a set of $r = N - n$ homogeneous generators for $I(X)$, then the degrees $d_1, \dots, d_r$ depend only on $X \subseteq \mathbb{P}^N$ and not on the choice of generators. This follows, for example, by noting that the Hilbert series of X (cf. [34, Ex. 7.15, p. 350]) is given by

$$\mathcal{H}_X(T) := \sum_{m=0}^{\infty} (\dim_k R_m) T^m = \frac{(1 - T^{d_1})(1 - T^{d_2}) \cdots (1 - T^{d_r})}{(1 - T)^{N+1}} \tag{18}$$

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[8] Actually, by a *variety* we mean an irreducible variety, but still, the word *irreducible* is sometimes added for emphasis. More precisely, by a *projective variety* defined over a field k, we mean a closed subscheme of $\mathbb{P}_k^N$ which is geometrically integral.

where R denotes the homogeneous coordinate ring of X and R_m its mth homogeneous component. It is customary to arrange $d_1, \ldots, d_r$ such that $d_1 \geq d_2 \geq \cdots \geq d_r$, and call the r-tuple $\mathbf{d} = (d_1, \ldots, d_r)$ the *multidegree* of X. Notice that an easy consequence of (18) is that the degree of X equals the product $d_1 d_2 \cdots d_r$.

The cohomology spaces of nonsingular complete intersections have a particularly simple structure, at least for all except one. Since we are mainly interested in varieties over finite field, we let $k = \mathbb{F}_q$ and consider étale ℓ-adic cohomology where ℓ is a prime different from $p = \operatorname{char} \mathbb{F}_q$.

Now if X is a nonsingular complete intersection in $\mathbb{P}^N$ defined over $\mathbb{F}_q$ and of dimension n, then for $0 \leq i \leq 2n$, we have

$$H^i(\bar{X}, \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell(-i/2) & \text{if } i \text{ is even and } i \neq n \\ 0 & \text{if } i \text{ is odd and } i \neq n. \end{cases} \tag{19}$$

Here $\mathbb{Q}_\ell(-i/2)$ denotes the 1-dimensional space $\mathbb{Q}_\ell$ with a 'twist' by the integer $-i/2$; the twist signifies that the Frobenius endomorphism[9] acts on this space as multiplication by $q^{i/2}$. At any rate, the Betti numbers $b_{i,\ell}(\bar{X})$ are either 1 or 0 according as i is even or odd, except if $i = n$. As for the middle cohomology $H^n(\bar{X}, \mathbb{Q}_\ell)$, it can be shown that its dimension, namely, $b_{i,\ell}(\bar{X})$ depends only on n, N and the multidegree $\mathbf{d}$. Thus we denote this by $b_n(N, \mathbf{d})$. Also we let

$$b'_n(N, \mathbf{d}) = \begin{cases} b_n(N, \mathbf{d}) - 1 & \text{if } n \text{ is even} \\ b_n(N, \mathbf{d}) & \text{if } n \text{ is odd.} \end{cases}$$

We call $b'_n(N, \mathbf{d})$ the *primitive* nth *Betti number* of the nonsingular complete intersection X in $\mathbb{P}^N$ of dimension n and multidegree $\mathbf{d}$. This primitive Betti number is given explicitly by the following formula.

$$b'_n(N, \mathbf{d}) = (-1)^{n+1}(n+1) + \sum_{c=r}^{N} (-1)^{N+c} \binom{N+1}{c+1} \sum_{\nu \in M(c)} \mathbf{d}^\nu, \tag{20}$$

where $M(c)$ denotes the set of all r-tuples $\nu = (\nu_1, \ldots, \nu_r)$ of positive integers such that $\nu_1 + \cdots + \nu_r = c$, and for any such tuple ν, $\mathbf{d}^\nu$ denotes the product $d_1^{\nu_1} \cdots d_r^{\nu_r}$, and where $r = N - n$, as before.

We remark that the formula (20) is equivalent to the following formula for the Euler characteristic of a nonsingular complete intersection X in $\mathbb{P}^N$ of dimension n and multidegree $\mathbf{d} = (d_1, \ldots, d_r)$:

$$\chi(X) := \sum_{i=0}^{2n} (-1)^i b_i(X) = d \sum_{c=0}^{n} (-1)^c \binom{N+1}{c+r+1} \sum_{\nu \in M_0(c)} \mathbf{d}^\nu, \tag{21}$$

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[9] For technical reasons, we consider here (and hereafter) the *geometric Frobenius* which is induced not by the map $x \mapsto x^q$ but by its inverse.

where $M_0(c)$ denotes the set of all r-tuples $\nu = (\nu_1, \dots, \nu_r)$ of nonnegative integers such that $\nu_1 + \cdots + \nu_r = c$, and d denotes the product $d_1 \cdots d_r$. The equivalence of (20) and (21) follows from (19), which gives the relation $\chi(X) = n + 1 + (-1)^n b'_n(N, \mathbf{d})$. In fact, the genesis of (20) and (21) is in a classical formula of Hirzebruch [14, Satz 2.4], which shows that for smooth complete intersections V_n in $\mathbb{P}^N_{\mathbb{C}}$ of dimension $n = N - r$, the generating function for the Euler characteristics is given as follows.

$$\sum_{n=0}^{\infty} \chi(V_n) T^n = \frac{d_1 \cdots d_r}{(1-T)^2} \prod_{i=1}^{r} \frac{1}{1 + (d_i - 1)T}. \tag{22}$$

It is an elementary exercise to derive (20) and/or (21) from (22) when X is obtained by 'reduction mod p' of a smooth complete intersection in characteristic zero. To pass from the complex case to positive characteristic, one appeals to an appropriate Comparison Theorem (see, for example, [28, Rem. 4.20]).

The formula for $b'_n(N, \mathbf{d})$ given above is rather complicated, and in applications, it is sometimes necessary to quickly find an estimate for this number. To this end, some bounds for $b'_n(N, \mathbf{d})$ are obtained in [10]. First, we note that there is the following simple bound:

$$b'_n(N, \mathbf{d}) \leq \binom{N+1}{n} (\delta + 1)^N \quad \text{where} \quad \delta = \max\{d_1, \dots, d_r\}.$$

A more precise, but slightly complicated, bound is the following.

$$b'_n(N, \mathbf{d}) \leq (-1)^{n+1}(n+1) + d \binom{N+1}{n} (\delta + 1)^n,$$

where δ is as above and d denotes the product $d_1 \cdots d_r$. In some special cases, we can work directly with the explicit formula and get a bound which is better than the two bounds above. For example, if $n = 1$ and d is as above, then it can be seen that

$$b'_1(N, \mathbf{d}) \leq (d-1)(d-2), \text{ with equality } \Leftrightarrow N = 2. \tag{23}$$

In other words, the equality holds if and only if X is a plane curve.

From the explicit formula and the bounds for $b'_n = b'_n(N, \mathbf{d})$, it is clear that Deligne's inequality (10) can be effectively used in applications. There is, however, the hypothesis that X has to be nonsingular. The following result from [10] extends Deligne's inequality to the case of complete intersections that are possibly singular. We use here the notion of the singular locus $\operatorname{Sing} X$ of a variety X. Its dimension $s = \dim \operatorname{Sing} X$ measures how far X is from being singular. If X is nonsingular, $\operatorname{Sing} X$ is empty and $s = -1$, by convention. If, for example, X is normal, then $s \leq n - 2$. In general, if X is an irreducible projective variety of dimension n, then $-1 \leq s \leq n - 1$.

Theorem 3.1. *Let X be an irreducible complete intersection of dimension n in $\mathbb{P}^N_{\mathbb{F}_q}$, defined by $r = N - n$ equations, with multidegree $\mathbf{d} = (d_1, \dots, d_r)$, and let s be an integer such that* $\dim \operatorname{Sing} X \le s \le n - 1$. *Then*

$$||X(\mathbb{F}_q)| - \pi_n| \le b'_{n-s-1}(N - s - 1, \mathbf{d})\, q^{(n+s+1)/2} + C_s(X) q^{(n+s)/2},$$

where $C_s(X)$ is a constant independent of q. If X is nonsingular, then $C_{-1}(X) = 0$. If $s \ge 0$, then

$$C_s(X) \le 9 \times 2^r \times (r\delta + 3)^{N+1} \quad \textit{where} \quad \delta = \max\{d_1, \dots, d_r\}.$$

The above theorem has some interesting corollaries, which may be worth noting. Let us mention them briefly.

For any irreducible complete intersection, we can take $s = n - 1$, and since $b'_0 = b_0 - 1 = 0$, the result implies that

$$||X(\mathbb{F}_q)| - \pi_n| = O(q^{n-(1/2)}). \tag{24}$$

This may be viewed as a weak version of the Lang-Weil inequality (6). Further, if X happens to be normal, then we can take $s = n - 2$, and now Theorem 3.1 yields

$$||X(\mathbb{F}_q) - \pi_n| \le b'_1(N - n + 1, \mathbf{d}) q^{n-(1/2)} + C_{n-2}(X) q^{n-1}.$$

This is a stronger result than the Lang-Weil inequality (6) because using (23), we see that $b'_1(N - n + 1, \mathbf{d}) \le (d - 1)(d - 2)$, with equality if and only if X is a hypersurface. On the other hand, when $s = -1$, the result is the same as Deligne's inequality (10). Thus, the inequality in Theorem 3.1 may be viewed as a common refinement of Deligne's inequality and the Lang-Weil inequality, at least for normal complete intersections. Other corollaries of Theorem 3.1 include a result of Aubry and Perret [4] on the number of points of singular curves, and a result of Shparlinskiĭ and Skorobogatov [38] on the number of points of complete intersections with at most isolated singularities. For the latter, one needs also an additional hypothesis about resolution of singularities. Finally, we remark that the inequality in Theorem 3.1 may be viewed as a more precise version of the estimate about the order of $|X(\mathbb{F}_q)|$, which was obtained by Hooley and Katz [15] in 1991.

The proof given in [10] of Theorem 3.1 can be very briefly sketched as follows. First, we use a variant of Bertini's Theorem to successively construct good hyperplane sections of X. In this way, we obtain a nonsingular complete intersection as a linear section of X. To deal with the corresponding cohomology spaces, we need a suitable generalization of the Weak Lefschetz Theorem for singular varieties. This is obtained using some work of Skorobogatov [39]. These ingredients together with the fundamental tools, namely (9) and (11), and also some recent work of Katz [19] yield the estimates in Theorem 3.1.

4. On the Lang-Weil Inequality

We have seen that if X is a smooth complete intersection of dimension n, defined over $\mathbb{F}_q$, then the difference between $|X(\mathbb{F}_q)|$ and the number $\pi_n = |\mathbb{P}^n(\mathbb{F}_q)|$ is of the order of $q^{n/2}$, thanks to Deligne's inequality (10). This estimate is the same as that given by Weil's inequality (4) or the Lang-Weil inequality (6) when $n = 1$. But it is significantly better than (6) when $n > 1$. In the case of general complete intersections with a singular locus of dimension s, the above difference is of the order of $q^{(n+s+1)/2}$, thanks to the Hooley-Katz estimate or its more precise version stated in the previous section. Again, this is usually much better than (6) when $n > 1$. However, for general projective varieties, essentially the only estimate that we have at our disposal is (24), which is given by the Lang-Weil inequality (6). Recall that for X in $\mathbb{P}^N$ with $\deg X = d$, the Lang-Weil inequality (6) states that

$$||X(\mathbb{F}_q)| - \pi_n| \leq (d-1)(d-2)q^{n-(1/2)} + Cq^{n-1},$$

where C is a constant which is independent of q (and depends only on N, n and d). A possible drawback in this result, particularly from the viewpoint of practical applications, is that almost no information is available about the size of C other than the fact that it is constant with respect to q. This situation is partially remedied by the following 'effective version' of the Lang-Weil inequality which is proved in [10].

We first make a preliminary definition. A projective algebraic variety X in the projective space $\mathbb{P}_k^N$ over a field k is said to be of *type* $(m, N, \mathbf{d})$, with $\mathbf{d} = (d_1, \ldots, d_m)$, if X can be defined by the vanishing of m homogeneous polynomials, in $N+1$ variables with coefficients in k, of degrees $d_1, \ldots, d_m$. Likewise, an affine algebraic variety X in $\mathbb{A}_k^N$ over a field k is said to be of *type* $(m, N, \mathbf{d})$, with $\mathbf{d} = (d_1, \ldots, d_m)$, if X can be defined by the vanishing of m polynomials, in N variables with coefficients in k, of degrees $d_1, \ldots, d_m$.

Theorem 4.1. *Suppose X is a projective variety in $\mathbb{P}_{\mathbb{F}_q}^N$ or an affine variety in $\mathbb{A}_{\mathbb{F}_q}^N$ defined over $\mathbb{F}_q$. Let $n = \dim X$ and $d = \deg X$. Then*

$$||X(\mathbb{F}_q)| - \pi_n| \leq (d-1)(d-2)q^{n-(1/2)} + C_+(\bar{X})\, q^{n-1},$$

where $C_+(\bar{X})$ is independent of q. Moreover if X is of type $(m, N, \mathbf{d})$, with $\mathbf{d} = (d_1, \ldots, d_m)$, and if $\delta = \max\{d_1, \ldots, d_m\}$, then we have

$$C_+(\bar{X}) \leq \begin{cases} 9 \times 2^m \times (m\delta + 3)^{N+1} & \text{if } X \text{ is projective} \\ 6 \times 2^m \times (m\delta + 3)^{N+1} & \text{if } X \text{ is affine.} \end{cases}$$

It may be noted that the statement is applicable to projective as well as affine varieties. In fact, the proof given in [10] is different from the one in the original paper of [24] and has the advantage of being applicable to varieties that are affine or projective apart from yielding an explicit bound for the constant C appearing in (6).

The fact that Theorem 4.1 is applicable to affine varieties gives as a corollary, the following lower bound for the number of points on a hypersurface H in $\mathbb{A}^N_{\mathbb{F}_q}$ defined by the vanishing of a polynomial in $\mathbb{F}_q[T_1, \ldots, T_N]$ which is irreducible over $\bar{\mathbb{F}}_q$:

$$|H(\mathbb{F}_q)| \geq q^{N-1} - (d-1)(d-2)q^{N-(3/2)} - 12(d+3)^{N+1}q^{N-2}.$$

This is analogous to the result of Schmidt [35], which gives a lower bound for $|H(\mathbb{F}_q)|$. Actually, in the lower bound of Schmidt, one is able to replace $12(d+3)^{N+1}$ by a much better constant, namely $6d^2$, but Schmidt's bound is only valid for large values of q.

In the remainder of this section, we will briefly outline the proof in [10] of Theorem 4.1. In fact, the basic idea is quite simple and natural. First of all, since X is irreducible, we have $H^{2n}(\bar{X}, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell(-n)$, which implies that $b_{2n,\ell} = 1$ and the only eigenvalue of $F|H^{2n}(\bar{X}, \mathbb{Q}_\ell)$ is $\omega_{2n\,1,\ell} = q^{2n/2} = q^n$. Next comes $H^{2n-1}(\bar{X}, \mathbb{Q}_\ell)$ and by Deligne's Main Theorem (11), this can be split into two parts as follows.

$$H^{2n-1}(\bar{X}, \mathbb{Q}_\ell) = H^{2n-1}_{+}(\bar{X}, \mathbb{Q}_\ell) \oplus H^{2n-1}_{-}(\bar{X}, \mathbb{Q}_\ell),$$

where $H^{2n-1}_{+}(\bar{X}, \mathbb{Q}_\ell)$ is the subspace spanned by the eigenvectors of F corresponding to the eigenvalues $\omega_{2n-1\,j,\ell}$ that are pure of weight $2n-1$, while $H^{2n-1}_{-}(\bar{X}, \mathbb{Q}_\ell)$ is an orthogonal complement of $H^{2n-1}_{+}(\bar{X}, \mathbb{Q}_\ell)$ in $H^{2n-1}(\bar{X}, \mathbb{Q}_\ell)$ (as a $\mathbb{Q}_\ell$-vector space). Following Serre, we call the $\mathbb{Q}_\ell$-vector space dimension of $H^{2n-1}_{+}(\bar{X}, \mathbb{Q}_\ell)$ the $(2n-1)$th *virtual Betti number* of X and denote it by $b^+_{2n-1,\ell}(\bar{X})$. Also, we let $b^-_{2n-1,\ell}(\bar{X}) = \dim H^{2n-1}_{-}(\bar{X}, \mathbb{Q}_\ell)$. Now it is clear from (9) and (11) that

$$||X(\mathbb{F}_q)| - q^n| \leq b^+_{2n-1,\ell}(\bar{X})q^{(n-\frac{1}{2})} + \left(b^-_{2n-1,\ell}(\bar{X}) + \sum_{i=0}^{2n-1} b^+_{i,\ell}(\bar{X})\right) q^{n-1}.$$

Using arguments similar to those in Section 1 following Weil's inequality (4), we see that the virtual $(2n-1)$th Betti number $b^+_{2n-1,\ell}(\bar{X})$ is, in fact, the best possible constant for the above inequality to hold (not only for q but for all its powers). Moreover, we can show that $b^+_{2n-1,\ell}(\bar{X})$ is a birational invariant and is independent of ℓ. Now, to complete the proof of Theorem 4.1, we have to bound $b^+_{2n-1,\ell}(\bar{X})$ in terms of the degree, and to effectively bound sums of Betti numbers such as those appearing in large parentheses in the above inequality. The former is done in two stages: first, by finding a 'typical curve' Y on X through degree-preserving linear sections in such a way that $b^+_{2n-1,\ell}(\bar{X}) \leq b^+_1(\bar{Y})$, and next, by showing that if Y is a curve of degree d, then $b^+_1(\bar{Y}) \leq (d-1)(d-2)$. The latter task of effectively bounding sums of Betti numbers is facilitated by some recent results of Katz [19] that were also used in the proof of Theorem 4.1. These results of Katz are, in fact, analogues of some classical results of Milnor [30] which show, for example, that if V is a complex affine algebraic variety of type $(m, N, \mathbf{d})$, with $\mathbf{d} = (d_1, \ldots, d_m)$, then $\sum b_i(V) \leq \delta(2\delta - 1)^{2N-1}$, where, as before, $\delta = \max\{d_1, \ldots, d_m\}$.

5. Conjectural Statements of Lang and Weil

An *abelian variety* is a projective variety, which is also a commutative group in such a way that the group operation and the map taking an element to its inverse are algebraic (morphisms). A classic example is given by an elliptic curve, where the points satisfy a group law. The work of Abel and Jacobi in the nineteenth century on elliptic integrals, and the more general abelian integrals, lead to the construction of an abelian variety, called the *Jacobian variety*, associated to any (smooth projective) curve. The Jacobian variety of a curve has dimension equal to the genus of that curve[10]. To any higher dimensional variety, one can associate certain abelian varieties known as the *Picard* and *Albanese* varieties, which play a role analogous to that of the Jacobian variety of a curve. For general projective varieties X over an arbitrary field, there are in fact, two distinct constructions of the Picard variety, namely, the Picard-Weil variety $\mathrm{Pic}_w X$ and the Picard-Serre variety $\mathrm{Pic}_s X$. Likewise, there is the Albanese-Weil variety $\mathrm{Alb}_w X$ and the Albanese-Serre variety $\mathrm{Alb}_s X$. When X is nonsingular, the two constructions coincide. If X is normal, then $\mathrm{Pic}_w X$ is the dual of $\mathrm{Alb}_w X$ while $\mathrm{Pic}_s X$ is the dual of $\mathrm{Alb}_s X$. For precise definitions of the Picard and Albanese varieties and some basic properties, we refer to [10] and the references therein. General references on abelian varieties are the books of Lang [23] and Mumford [33].

Now suppose X is an irreducible projective variety in $\mathbb{P}^N$ defined over $\mathbb{F}_q$. Let n be the dimension of X and d be the degree of X. For simplicity, we assume that X has at least one $\mathbb{F}_q$-rational point (which can be ensured by passing, if necessary, to a finite extension of $\mathbb{F}_q$).

When Lang and Weil [24] proved the inequality (6), namely,

$$||X(\mathbb{F}_q)| - \pi_n| \leq (d-1)(d-2)q^{n-(1/2)} + Cq^{n-1},$$

they showed in the same paper that if K is an algebraic function field of dimension n over $k = \mathbb{F}_q$, then there is a constant γ for which (6) holds with $(d-1)(d-2)$ replaced by γ, for any model X of K/k, and moreover, the smallest such constant γ is a birational invariant. Subsequently, Lang and Weil went on to conjecture that this constant γ can be described *algebraically* as being twice the dimension of the associated Picard variety, at least when X is nonsingular. Notice that when X is a smooth curve of genus g, then from the remarks made in Section 1 following Weil's inequality (4), we see that $\gamma = 2g$, and thus the conjecture is verified in this case.

The conjectures of Lang and Weil preceded an analysis of the zeros and poles of the zeta function of X. They showed that the zeros and poles of $Z(X, T)$ in the open disc $|T| < q^{-(n-1)}$ are birational invariants, and that in the smaller disc $|T| < q^{-(n-1/2)}$ there is exactly one pole of order 1 at $T = q^{-n}$ (see Fig. 1). Then they wrote: *about the behavior of* $Z(X, T)$ *for* $|T| \geq q^{-(n-1/2)}$, *we can only*

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[10] For a lucid exposition of basic facts concerning Jacobian varieties, in the classical case of complex curves, see the China lectures of Griffiths [11, Ch. V.].

make the following conjectural statements, which complement the conjectures of Weil. In addition to the conjecture mentioned above about γ, these *conjectural statements* are to the effect that when X is projective and nonsingular, the quotient

$$\frac{Z(X,T)(1-q^nT)}{f_c(P,T)} \tag{25}$$

has no zeros or poles inside $|T| < q^{-(n-1)}$ and at least one pole on $|T| = q^{-(n-1)}$, where P denotes the Picard variety of X, and $f_c(P,T)$ the 'characteristic polynomial' of P.

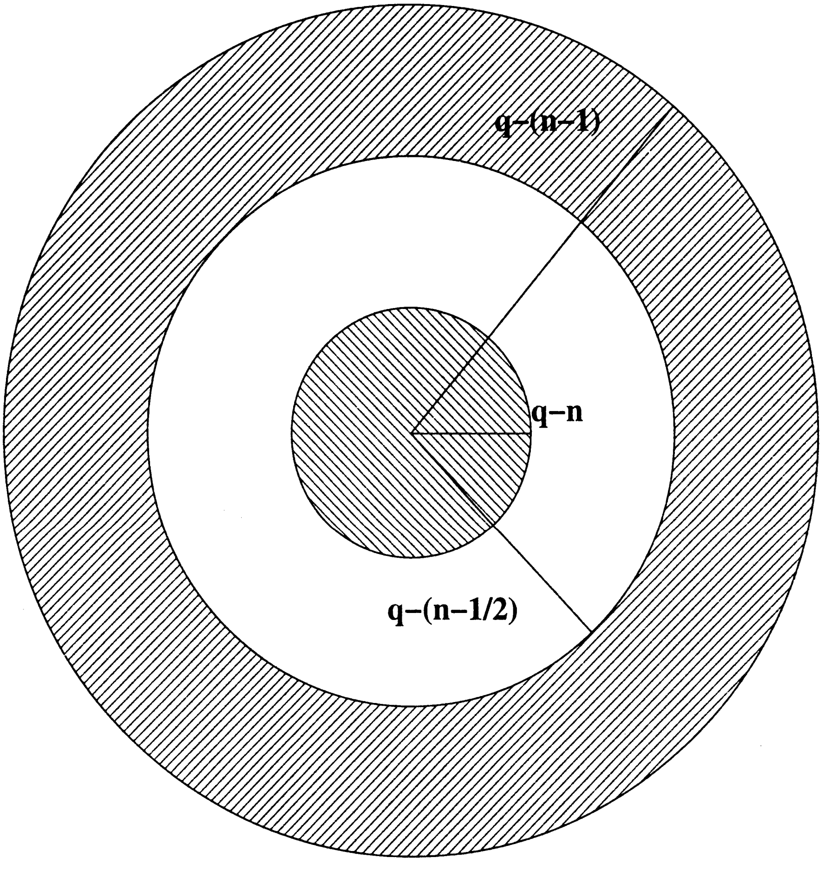

Figure 1 Disks in which the zeros and poles of the zeta function $Z(X,T)$ were analyzed by Lang and Weil

The characteristic polynomial of an abelian variety A over $\mathbb{F}_q$ can be defined as follows. Choose a prime $\ell \neq p = \operatorname{char} \mathbb{F}_q$. Let $T_\ell(A)$ denote the *Tate module* of A, which is the projective limit of $\{A_{\ell^n} : n \geq 1\}$ with respect to the maps induced by multiplication by ℓ (for $m \in \mathbb{Z}$, we let $A_m = \{a \in A(\overline{\mathbb{F}}_q) : ma = 0\}$ be the set of m-torsion points of A). If $\dim A = g$, then the Tate module $T_\ell(A)$ is a free $\mathbb{Z}_\ell$-module of rank $2g$; consequently, $V_\ell(A) := T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is a $\mathbb{Q}_\ell$-vector space of dimension $2g$. Now if $\varphi : \overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q$ is the Frobenius map given by $\varphi(x) = x^q$, then φ induces an endomorphism of $T_\ell(A)$, which we denote by $T_\ell(\varphi)$. The *characteristic polynomial* $f_c(A, T)$ of A is defined to be the characteristic polynomial of the linear map $T_\ell(\varphi)$ of $T_\ell(A)$; in other words,

$$f_c(A, T) := \det(T - T_\ell(\varphi)) = T^{2g} - a_1 T^{2g-1} + \cdots + (-1)^{2g} a_{2g},$$

where $a_1, \ldots, a_{2g}$ are in $\mathbb{Z}_\ell$. We call a_1 to be the *trace* of φ and denote it by $\mathrm{Tr}(\varphi)$. The constant coefficient a_{2g} is sometimes called the *degree* of φ and denoted by $\deg \varphi$.

We are now in a position to describe some of the main results from [10] in connection with the conjectural statements of Lang and Weil. To begin with, let us note that from the discussion in the last paragraph of the previous section, it is clear that the birational invariant γ that Lang and Weil talked about is nothing but the virtual $(2n-1)$th Betti number $b^+_{2n-1,\ell}(\bar{X})$. Also, it is clear from the factorization (8) and Deligne's Main Theorem (11) (or the Riemann hypothesis, in the smooth case), that to prove the assertion of Lang and Weil concerning the quotient (25), it suffices to relate the polynomial $f_c(P, T)$ with $P_{2n-1,\ell}(X, T)$. The following result from [10] gives, in fact, a relationship between the corresponding algebraic objects. Note that the hypothesis of this result is fulfilled when X is nonsingular.

Proposition 5.1. *Suppose X is a normal projective variety defined over $\mathbb{F}_q$ and of dimension $n \geq 2$ such that X is also regular in codimension 2. Then the $\mathbb{Q}_\ell$ vector spaces $V_\ell(\mathrm{Alb}_w X)$ and $H^{2n-1}(\bar{X}, \mathbb{Q}_\ell(n))$ are isomorphic (in fact, by a $\mathfrak{g}$-equivariant isomorphism, where $\mathfrak{g} = \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ is the absolute Galois group of $\mathbb{F}_q$). As a consequence,*

$$b_{2n-1,\ell}(\bar{X}) = 2 \dim \mathrm{Alb}_w X = 2 \dim \mathrm{Pic}_w X.$$

In particular, the penultimate Betti number is even and independent of ℓ.

To extend this result to varieties that are not necessarily regular in codimension 2, we need a weak version of resolution of singularities, namely the condition on X that X *is birationally equivalent to a normal projective variety $\tilde{X}$ defined over k, which is regular in codimension* 2. We will refer to this condition as (**RS2**).

Theorem 5.2. *Suppose X is a projective variety defined over $\mathbb{F}_q$ and of dimension $n \geq 2$ which satisfies* (**RS2**). *Then the $\mathbb{Q}_\ell$ vector spaces $V_\ell(\mathrm{Alb}_w X)$ and $H^{2n-1}_+(\bar{X}, \mathbb{Q}_\ell(n))$ are isomorphic (in fact, by a $\mathfrak{g}$-equivariant isomorphism, where $\mathfrak{g} = \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ is the absolute Galois group of $\mathbb{F}_q$). As a consequence, $b^+_{2n-1,\ell}(\bar{X}) = 2 \dim \mathrm{Alb}_w X$.*

If X is nonsingular, one has Poincaré duality for the étale cohomology space and Proposition 5.1 would imply a similar result for H^1 and the first Betti number $b_1(\bar{X})$. But for a general X, this is not true. However, it may be interesting to note the following result in [10], which involves the Serre construction of the Picard or the Albanese varieties. This result can be used, in turn, to obtain a weak version of Poincaré duality when the hypothesis of Proposition 5.1 is met.

Proposition 5.3. *Suppose X is a normal projective variety defined over $\mathbb{F}_q$ and of dimension n. Then the $\mathbb{Q}_\ell$ vector spaces $V_\ell(\mathrm{Pic}_s X)$ and $H^1(\bar{X}, \mathbb{Q}_\ell)$ are isomorphic (in fact, by a $\mathfrak{g}$-equivariant isomorphism, where $\mathfrak{g} = \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$, provided we twist $V_\ell(\mathrm{Pic}_s X)$ by -1). As a consequence, $b_{1,\ell}(\bar{X}) = 2\dim \mathrm{Pic}_s X$; in particular, the first Betti number is even and independent of ℓ.*

Recall that the space $H_+^{2n-1}(\bar{X}, \mathbb{Q}_\ell)$ and its dimension, the $(2n-1)$th virtual Betti number, $b_{2n-1,\ell}^+(X)$ were defined in the previous section. Let us also define $P_{2n-1,\ell}^+(X, T) = \det(1 - TF|H_+^{2n-1}(\bar{X}, \mathbb{Q}_\ell))$.

Theorem 5.4. *Let X be any variety defined over $\mathbb{F}_q$ of dimension $n \geq 2$, and let $g = \dim \mathrm{Alb}_w X$. Then*

$$P_{2n-1,\ell}^+(X, T) = q^{-g} f_c(\mathrm{Alb}_w X, q^n T).$$

In particular,

$$b_{2n-1,\ell}^+(X) = 2g \quad \textit{and} \quad Tr(F \mid H_+^{2n-1}(\bar{X}, \mathbb{Q}_\ell)) = q^{n-1}\,\mathrm{Tr}\,(\varphi).$$

If X is normal, then $\mathrm{Pic}_w X$ and $\mathrm{Alb}_w X$ are the duals of each other, and $f_c(\mathrm{Alb}_w X, T) = f_c(\mathrm{Pic}_w X, T)$. Thus, in view of the remarks made earlier in this section, the conjectural statements of Lang and Weil follow as a particular case of Theorem 5.4.

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On An Additive Question

S. Srinivasan

Let $k > 0$ be a given integer. Here we obtain some results concerning solvability of $\mathbb{A} \oplus \mathbb{B} = \mathbb{Z}^k$, in $\mathbb{B}$, with respect to a finite set $\mathbb{A}$ of a given 'diameter'. And also announce some other results regarding a conjecture from R. Tijdeman in the case $k = 1$.

1. Introduction

For given sets A, B of (- say,) integers, write $C = A + B$ to denote the set of all integers c representable as $c = a + b$ with $a \in A$ and $b \in B$. For many 'inverse' problems the theme is to seek 'structural' informations concerning A and B when certain properties of C are known. In the present context, the available information is that every member c of C has a *unique* representation $c = a + b$. Such A and B are called 'complementing C-pairs', and then we use the notation $C = A \oplus B$. We use $| A |$ to denote the cardinality of A, and by 'diameter' of A is meant the maximum of the absolute value of differences between any two members of A.

It seems the case $C = \mathbb{Z}$, was first encountered by de Bruijn in 1950, while studying 'Bases' for the set of integers. In 1974, Swenson [6] showed that any two finite sets of integers A and B, such that all sums $a + b$ $(a \in A, b \in B)$ are distinct, can be extended to two infinite complementing $\mathbb{Z}$-pairs. Thus there is no effective characterisation of all complementing $\mathbb{Z}$-pairs. However, we can consider the problem when A is a given finite set, to ask if there are complementing sets B for it. Here are two 'directions'; namely, (*i*) when $| A |$ is fixed, or (*ii*) when diameter of A is fixed. In this article we study (*ii*), in the more general case of $C = \mathbb{Z}^k$ (with a positive integer k being given).

As for (*i*), de Bruijn had considered the situation where $| A | = h$ is a prime, and formulated a conjecture which would imply a certain structure for what he called A-bases. This conjecture was proved by Sands [4] in 1957. More recently, R. Tijdeman gave another proof of it, and also enunicated an extension of this conjecture to the case when h is *not* necessarily a prime (see, p. 226 of [7]). This latter conjecture can be expressed equivalently as follows.

Conjecture. T (de Bruijn-Tijdeman). If $\mathbb{Z} = A \oplus B$, $| A | < \infty$, then either A or B is contained in an arithmetic progression with common difference exceeding unity.

In view of some counter-examples, it has been known that this conjecture is not true in its generality. For more literature on this topic the reader is referred also to

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2000 Mathematics Subject Classification: 11B75.

[1], [2], [3] and [5]. In Section 5, we announce some results noted in connection with this conjecture.

2. Statement of Results

Let (π) denote the property that for a given finite set $\mathbf{A}(\subseteq \mathbb{Z}^k)$, there exists a $\mathbf{B}$ such that $\mathbf{A}\oplus\mathbf{B} = \mathbb{Z}^k$; briefly expressed as $\mathbf{A} \in (\pi)$. Here we study the question of 'decidability' of $\mathbf{A} \in (\pi)$, and use the phrase 'in proper time' to mean 'taking time not exceeding a constant depending at most on k and the diameter of $\mathbf{A}$'. [Here the phrase 'time' should be regarded as meaning the 'number of successive basic operations involved to perform an algorithm under consideration'.] For example, our question can be reformulated as "Is it possible to determine whether $\mathbf{A} \in (\pi)$, in proper time?".

Before stating our results, we introduce some notation. For a value of k (specific to the context), let $\mathbf{O}$ and I denote points (in $\mathbb{Z}^k$) with all components equal, respectively, to 0 and 1. And for $1 \leq j \leq k$, let $e(j)$ denote the point (in $\mathbb{Z}^k$) having the j^{th} component $= 1$, while all others $= 0$.

Theorem 1. *Let* $\mathbf{A} \subseteq \mathbb{Z}^k$ *be a given (non-empty) finite set. Then we can determine, in proper time, a finite set* F *and an associated set* $G \subseteq F^{k+1}$, *so that we will have* $\mathbf{A} \in (\pi)$ *if and only if there is a map* f *from* $\mathbb{Z}^k$ *into* F *such that*

$$(f(P), f(P+e(1)), \dots, f(P+e(k))) \in G, \tag{1}$$

for all $P \in \mathbb{Z}^k$.

In connection with the existence of a map f specified in Theorem 1, we have

Theorem 2. *Let* $k > 1$. *Suppose that sets* F_k *and* $G_k \subseteq F_k^{k+1}$ *are given. Then sets* F_{k-1} *and* $G_{k-1} \subseteq F_{k-1}^k$ *can be defined such that there is a map* f_k *satisfying* (1) *with* $F = F_k$, $G = G_k$ *and* $f = f_k$, *if and only if there is a map* f_{k-1} *satisfying* (1) *with* (k *replaced by* $k-1$), $F = F_{k-1}$, $G = G_{k-1}$ *and* $f = f_{k-1}$.

Although Theorem 2 reduces, theoretically, the question of existence of f in Theorem 1 to the case $k = 1$, its applicability (in manner of proof here) in proper time is limited, since F_{k-1} need not necessarily be finite. Hence the question of $\mathbf{A} \in (\pi)$ to be decided in proper time, remains not completely resolved.

3. Proof of Theorem 1

To start with, we have some notation. Since $\mathbf{A} \in (\pi)$ is unaffected on translation of $\mathbf{A}$ by a fixed point, it can be presumed that each point in $\mathbf{A}$ has all components non-negative. So, we assume that

$$\mathbf{A} \subseteq \prod_{j=1}^{k} [0, N_j) (\subseteq \mathbb{Z}^k), \tag{2}$$

with some given positive integers N_j; $1 \leq j \leq k$. Then the phrase 'in proper time' would mean 'in time not exceeding a constant depending at most on k and N, where $N = N_1 \dots N_k$'. We show that we may further assume

$$\mathbf{O} \in \mathbf{A}. \tag{3}$$

For this set

$$T = \|t_{ji}\|; \, t_{ji} = 1, (i \leq j \leq k); \, t_{ji} = 0, (j < i \leq k), \tag{4}$$

where $\|t_{ji}\|$ stands for the matrix of order k. Then, in an obvious notation, for each integer $r \geq 1$ observe that

$$\mathbf{A} \in (\pi) \Leftrightarrow T^r\mathbf{A} \in (\pi); \; T^r := \|t_{ji}(r)\|, \tag{5}$$

with

$$t_{ji}(r) = \binom{r+j-i-1}{j-i} (i \leq j \leq k); \, t_{ji}(r) = 0, (j < i \leq k). \tag{6}$$

[Here T^r denotes the r-th power of the matrix T.]

For given $\mathbf{x} = (x_1, \dots, x_k)$ and $r \geq 1$, write

$$T^r\mathbf{x} := (x_1(r), \dots, x_k(r)),$$

so that

$$x_j(r) = \sum_{i=1}^{j} \binom{r+j-i-1}{j-i} x_i, \, (1 \leq j \leq k). \tag{7}$$

Now let $\mathbf{x}, \mathbf{x}'$ be two distinct points in $\mathbb{Z}^k$, and $1 \leq a \leq k$ be defined through

$$x_i = x_i'(1 \leq i < a), x_a > x_a' \text{ (say)}.$$

Then, for each $r \geq 1, x_j(r) = x_j'(r)$ for $1 \leq j < a$, and for $a \leq j \leq k$ we get, by (7),

$$x_j(r) - x_j'(r) = \sum_{i=a}^{j} \binom{r+j-i-1}{j-i} (x_i - x_i'),$$

which equals

$$(x_a - x_a') \binom{r+j-a-1}{j-a} + \theta M \sum_{i=a+1}^{j} \binom{r+j-i-1}{j-i},$$

with $M = \max_{1 \leq i \leq k}(|x_i| + |x_i'|)$ and some $|\theta| \leq 1$. Since the last sum here is $= \binom{r+j-a-1}{j-a-1} \leq \frac{k}{r} \binom{r+j-a-1}{j-a}$, we obtain for all $r \geq r_0(k, M)$

$$x_j(r) - x_j'(r) > 0, (a \leq j \leq k) \tag{8}$$

on using $x_a - x'_a \geq 1$. Then it is easily concluded from (8) that, for a suitable r, depending at most on k and N, the set $T^r(\mathbf{A})$ contains two points $\mathbf{y}, \mathbf{y}'$ such that

$$y_j \leq x_j \leq y'_j, \forall \mathbf{x} \in T^r\mathbf{A}, (1 \leq j \leq k).$$

This proves that (3) can be realised in proper time, and that

$$(3') \qquad (N_1 - 1, \ldots, N_k - 1) \in \mathbf{A},$$

can also be assumed.

Next, for $\mathbf{m} = (m_1, \ldots, m_k) \in \mathbb{Z}^k$ define

$$[\mathbf{m}] = \{\mathbf{x} \in \mathbb{Z}^k : m_j N_j \leq x_j < (m_j + 1)N_j; 1 \leq j \leq k\},$$

so that $[\mathbf{m}] = (m_1N_1, \ldots, m_kN_k) + [\mathbf{O}]$. Further, for $\mathbf{C} \subseteq \mathbb{Z}^k$, introduce

$$\mathbf{C}(m) \subseteq [\mathbf{O}]; [\mathbf{m}] \cap \mathbf{C} = (m_1N_1, \ldots, m_kN_k) + \mathbf{C}(m).$$

For $D \subseteq U := \{1, \ldots, k\}$, set $e(D) = \sum_{j \in D} e(j)$ so that $e(\phi) = \mathbf{0}$ and $e(U) = I$. Let $\mathbf{E}$ denote the collection of all subsets of $[\mathbf{O}]$, and $\mathbf{C}[P] := (\ldots, \mathbf{C}(P + e(D)), \ldots) \in \mathbf{E}^K$; $P \in \mathbb{Z}^k$, $K := 2^k$, where E^k denotes the 'cartesian product of E taken k times'. Now, if $\mathbf{A} \oplus \mathbf{B} = \mathbb{Z}^k$, we observe that the component $\mathbf{B}(P + I)$ in $\mathbf{B}[P]$ is *uniquely* determined by the other $(K - 1)$ components, in view of (3). In fact, this property enables us to extend it for all members of $\mathbf{E}^K$. Indeed, call a member of $\mathbf{E}^K$ considered as a $\mathbf{B}[\mathbf{0}]$ (say, for definiteness), $\mathbf{A}$-*admissible* if its U-component is determined uniquely by the others, assuming (3) and $\mathbf{A} \in (\pi)$. Let F denote the collection of all $\mathbf{A}$-*admissible* members of $\mathbf{E}^K$. Note that (generally), for $1 \leq j \leq k$, the D-component of $\mathbf{C}[P]$ with $D \ni j$ equals the $(D - \{j\})$-component of $\mathbf{C}[P + e(j)]$. Let $G \subseteq F^{k+1}$ denote the collection of all $(k + 1)$-tuples $(F(0), F(1), \ldots, F(k))$ where each component $F(j)$ has its $(D - \{j\})$-th component equal to the D-th component of $F(0)$, for all $D \ni j$ and for each $1 \leq j \leq k$. Obviously from given $\mathbf{A}$ one can determine F and G in proper time. Now, if we have $\mathbf{A} \oplus \mathbf{B} = \mathbb{Z}^k$, the map f can be defined by $f(P) = \mathbf{B}[P]$ which, in view of the above construction, belongs to F, and satisfies (1). Conversely, given a map f subject to (1), we can introduce $\mathbf{B}$ as the union of the ϕ-components of $f(\mathbf{m})$ translated by $(m_1N_1, \ldots, m_kN_k)$, taken over all $\mathbf{m} \in \mathbb{Z}^k$. Then it can be easily checked that $\mathbf{A} \oplus \mathbf{B} = \mathbb{Z}^k$.

The case of $k = 1$ is completely solved in proper time, because if $\mathbf{A} \in (\pi)$ then the sequence $\{f(n); n \in \mathbb{Z}\}$ would necessarily have to be periodic with a period depending only on N, in view of (1).

Thus in particular, $\mathbf{A} \notin (\pi)$ if $|G| = 0$, whereas if there is a member of G with all components equal then we can conclude that $\mathbf{A} \in (\pi)$. And the case $k = 1$ can be decided in proper time.

Remarks. Immediately we observe that, for the purpose of Theorem 1, one can take $F = \{1, 2, \ldots R\}$ and G as being determined correspondingly (in a trivially modified notation), where R denotes the cardinality of the set F mentioned in Theorem 1.

In the case $k = 1$, observe also that for any given $\mathbf{A}$ of 'diameter' N, there can only be (- if any -) a finite number (with a bound depending on N) of $\mathbb{Z}$-complementing sets $\mathbf{B}$, because $|E|$ is bounded in terms of N (and k). Also, in view of (3′), in the above proof we could *refine* the definition of '$\mathbf{A}$-admissibility' by noting in addition that $\mathbf{B}(P)$ is determined uniquely by the other components in $\mathbf{B}[P]$, thereby possibly reducing the size of F and hence of G, too. Further, we note that there is a (natural) bijection between the maps f, and the sets $\mathbf{B}$ (complementing the given set $\mathbf{A}$).

4. Proof of Theorem 2

To begin with, we note that the existence of f_k so that $\forall P \in \mathbb{Z}^k, (f_k(P), f_k(P+e(1)), \ldots, f_k(P+e(k))) \in G_k$ is equivalent to the existence of a (modified) f_k such that

$$(1') \qquad (f_k(P), f_k(P+e(1)), \ldots, f_k(P+e(k-1)), f_k(P+I)) \in G_k.$$

For this, observe that

$$\Delta(e(j)) = e(j), 1 \le j < k; \ \Delta(e(k)) = I,$$

where $\Delta = \|d_{ij}\|$ with $d_{ij} = 0$, except for $d_{ii} = 1 = d_{ik}$; $1 \le i \le k, 1 \le j < k$. And set the value of (modified) f_k at P as equal to that of (original) f_k at $\Delta^{-1}P$. Now, we define F_{k-1} as the collection of all sequences

$$\Phi = \{\varphi_n : \exists g_n = (\varphi_n, \tau_{n1}, \ldots, \tau_{nk-1}, \varphi_{n+1}) \in G_k; \forall n \in \mathbb{Z}\},$$

with some suitable τ's. Further, define G_{k-1} as the collection of $(\Phi_0, \cdots, \Phi_{k-1}) \in F_{k-1}^k$ with, for $0 \le j < k$,

$$(9) \qquad \Phi_j \equiv (\varphi_{nj}) : (\varphi_{n0}, \varphi_{n1}, \ldots, \varphi_{nk-1}, \varphi_{(n+1)0}) \in G_k,$$

for all $n \in \mathbb{Z}$.

If there is an f_k satisfying (1′), $\forall P \in \mathbb{Z}^k$, then f_{k-1} fulfilling

$$(1'') \qquad (f_{k-1}(P_0), f_{k-1}(P_0 + e(1)), \ldots, f_{k-1}(P_0 + e(k-1))) \in G_{k-1},$$

for all $P_0 \in \mathbb{Z}^{k-1}$, can be determined as follows. Take $P = (P_0, 0)$ and write $\varphi_{nj} = f_k(P + nI + e(j))$ to satisfy (9) in view of (1′). Then, it is easily seen that (1″) holds on setting $f_{k-1}(P_0) := \{\varphi_{n0} : n \in \mathbb{Z}\} \in F_{k-1}$. Likewise, starting with an f_{k-1} subject to (1″) we can determine f_k to have (1′) by reversing the argument. This completes the proof of Theorem 2.

Remark. Here we note that had the finiteness of F_k implied the finiteness of F_{k-1}, then one could conclude the decidability of $\mathbf{A} \in (\pi)$ in proper time. However, the proof shows that both the collections of f_k's and of f_{k-1}'s are with the same cardinality.

5. Concluding Remarks

Let $n(>1)$ be an integer and let $\underline{x} = (x_0, x_1, \ldots, x_{n-1})$ denote a vector with components from $\mathbb{Q}$, the set of rationals. Further, define V_1 as the collection of $\underline{x}$ with equal components, and for $d \neq 1, d \mid n$, we define V_d as the collection of $\underline{x}$ with components satisfying the following conditions:

(i) x_j's are periodic in j with period d, and

(ii) for each prime $p \mid d$ and any given residue class $b \pmod{d/p}$, we have $\sum' x_i = 0$ where $'$ denotes the condition $i \equiv b \pmod{d/p}$. Then, we have

Proposition 1. *There holds* $\mathbb{Q}^n = \oplus_{d|n} V_d$.

The next result can be considered as a (geometric) interpretation of Conjecture T. For its formulation we consider only vertices of the unit cube $\underline{x}$ with $x_j = 0$, or $= 1$, further restricted by the stipulation that for any given arithmetic progression of j's $(0 \leq j < n)$ (with common difference > 1), there exists a value of j, not in this a.p., such that $x_j = 1$. Now, we can state, in an obvious notation.

Proposition 2. *The following statement is equivalent to Conjecture T. For any given pair of vertices of the unit cube $\underline{u}, \underline{v}$, as described above satisfying $(\sum u_j)(\sum v_j) = n$, there is a divisor $d(>1)$ of n such that both the projections of $\underline{u}$ and $\underline{v}$ on V_d are non-zero.*

For the sake of simplicity we present our last result only in a qualitative form. Set, for squarefree $n, \underline{a} = (a_1, a_2, \ldots, a_{\varphi(n)})$, where $1 + \sum a_j x^j$ is the cyclotomic polynomial of order n. Then, we have

Proposition 3. *With respect to an explicit square matrix M, of order $\varphi(n)$, we have $\underline{a} = M\underline{w}$, where $\underline{w}$ has all its components $= 1$.*

The proofs of these Propositions are to be published elsewhere after some more work, in view of the statement of Proposition 2, so as to distinguish those cases when the conjecture could fail.

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n-Colour Partitions

A.K. Agarwal

1. Introduction

In this paper we shall survey the advances that have been made in recent years in dealing with problems on n-colour partitions. An n-colour partition (also called a partition with n copies of n) of a positive integer ν is a partition in which a part of size n, can come in n different colours denoted by subscripts: $n_1, n_2, \ldots, n_n$ and the parts satisfy the order.

$1_1 < 2_1 < 2_2 < 3_1 < 3_2 < 3_3 < 4_1 < 4_2 < 4_3 < 4_4 < 5_1 < \ldots.$

For example, there are 13 n-colour partitions of 4 with n copies of n:

$4_1, 4_2, 4_3, 4_4, 3_1 + 1_1, 3_2 + 1_1, 3_3 + 1_1, 2_2 + 2_1, 2_2 + 2_2, 2_1 + 2_1, 2_2 + 1_1 + 1_1, 2_1 + 1_1 + 1_1, 1_1 + 1_1 + 1_1 + 1_1$.

Now if $P(\nu)$ denotes the number of n-colour partitions of ν then by using the standard techniques of partition theory [14, Chap. 1] it can be easily shown that

$$1 + \sum_{\nu=1}^{\infty} P(\nu)q^{\nu} = \prod_{n=1}^{\infty}(1 - q^n)^{-n}. \tag{1.1}$$

P.A. MacMahon [24, p. 1421] proved that the right-hand side of (1.1) is indeed the generating function for plane partitions. A plane partition is an array of non-negative integers.

$$\begin{array}{|cccc}
\hline
n_{1,1} & n_{1,2} & n_{1,3} & \cdots \\
n_{2,1} & n_{2,2} & n_{2,3} & \cdots \\
\cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot &
\end{array}$$

for which $\sum_{i,j} n_{i,j} = n$ and the rows and columns are in non-increasing order: $n_{i,j} \geq n_{(i+1),j}, n_{i,j} \geq n_{i(j+1)}$, for all $i, j \geq 1$. The non-zero entries $n_{i,j} > 0$ are called the parts of the partition. A plane partition is symmetric if $n_{i,j} = n_{j,i}$ for

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all i and j. For clarity, we list the 13 plane partitions of 4

$$\begin{array}{llllllll} 4, & 31, & 3, & 22, & 2, & 211, & 21, & 2,\\ & & 1 & & 2 & & 1 & 1\\ & & & & & & & 1 \end{array}$$

$$\begin{array}{lllll} 1111, & 111, & 11, & 11, & 1.\\ & 1 & 1 & 11 & 1\\ & & 1 & & 1\\ & & & & 1 \end{array}$$

n-Colour partitions which arise in Baxter's solution of the hard hexagon model [15, Chap. 14] were initially studied by G.E. Andrews and A.K. Agarwal for giving combinatorial interpretations of several q-series identities. In fact before Andrews and Agarwal began studying n-colour partitions they (n-colour partitions) had been used indirectly in many studies of plane partitions. See for instance, Chaundy [17], Cheema and Gordon [18] and Sagan [27]. But they had never been studied in their own right. Now a full theory (almost parallel to the theory of the classical partitions) is being developed for them. We shall illustrate this in the following sections.

2. An n-Colour Analogue of Euler's Identity

In 1748 Euler [19] discovered the following:

Theorem 2.1. *The number of partitions of a positive integer ν into distinct parts is equal to the number of partitions of ν into odd parts.*

An n-colour analogue of Theorem 2.1 was recently given in [9] in the following form.

Theorem 2.2. *Let $A(\nu)$ denote the number of n-colour partitions of a positive integer ν into distinct parts. Let $B(\nu)$ denote the number of n-colour partitions of ν such that the even parts appear with even subscripts only. Then*

$$A(\nu) = B(\nu), \text{for all } \nu.$$

Example. $A(4) = 8$, Since the relevant partitions are:

$$\begin{array}{llll} 4_1, & 4_2, & 4_3, & 4_4,\\ 3_1 1_1, & 3_2 1_1, & 3_3 1_1, & \\ 2_1 2_2. & & & \end{array}$$

Also, $B(4) = 8$. In this case the relevant partitions are

$$\begin{array}{lll} 4_2, & 4_4, & \\ 3_1 1_1, & 3_2 1_1, & 3_3 1_1,\\ 2_2 2_2, & 2_2 1_1 1_1, & 1_1 1_1 1_1 1_1. \end{array}$$

Proof. From the standard techniques of partition theory we have:

$$\begin{aligned}\sum_{\nu=0}^{\infty} A(\nu)q^{\nu} &= \prod_{n=1}^{\infty}(1+q^n)^n \\ &= \prod_{n=1}^{\infty}\frac{(1-q^{2n})^n}{(1-q^n)^n} \\ &= \prod_{n=1}^{\infty}\frac{(1-q^{2n})^n}{(1-q^{2n})^{2n}(1-q^{2n-1})^{2n-1}} \\ &= \prod_{n=1}^{\infty}\frac{1}{(1-q^{2n})^n(1-q^{2n-1})^{2n-1}} \\ &= \sum_{n=1}^{\infty} B(\nu)q^{\nu}\end{aligned}$$

Since a power series expansion of a function is unique, we see that

$$A(\nu) = B(\nu), \text{for all } \nu.$$

□

Remark. The condition "even parts appear with even subscripts only" on the partitions enumerated by $B(\nu)$ can be replaced by the condition "even parts appear with odd subscripts only" in view of the fact that n-colour partitions where even parts appear with even subscripts only and the n-colour partitions where even parts appear with odd subscripts only are both generated by the same function.

$$\prod_{n=1}^{\infty}(1-q^{2n})^{-n}.$$

Several other restricted n-colour partition functions were also studied in [9].

3. An n-Colour Analogue of Sylvester's Theorem

In a graphical representation of a partition (called Ferrers graph), each part is represented by a horizontal row of dots. All rows are aligned on the left. For example, the Ferrers graph of the partition $\pi = 4 + 3 + 1$ of 8 is

0 0 0 0

0 0 0

0

The conjugate partition of a given partition is formed by reading the graph by columns. The conjugate partition of π (denoted by π^{c}), then, is

$$8 = 3 + 2 + 2 + 1.$$

A partition is called self-conjugate if it is identical with its conjugate. For example, $3 + 2 + 1$ is a self-conjugate partition of 6.

Many partition identities are easily proved by means of graphs. For example, the following two can be mentioned:

Theorem 3.1. *The number of partitions of n in which at most m parts appear equals the number of partitions of n in which no part exceeds m.*

Theorem 3.2. (Sylvester, [21, Th. 347]) *The number of partitions of ν with distinct odd parts equals the number of self-conjugate partitions of ν.*

We now give some definitions which are used in the statements of the n-colour analogues of Theorems 3.1 and 3.2.

Definition 3.1. (see [13]) Let $\pi = (a_1)_{b_1} + (a_2)_{b_2} + \cdots + (a_r)_{b_r}$ be an n-colour partition of ν. We call $(a_i)_{a_i - b_i + 1}$ the conjugate of $(a_i)_{b_i}$. An n-colour partition of ν obtained from π by replacing each of its parts by its conjugate will be called the conjugate of π and will be denoted by π^c. For example, if we consider $\pi = 5_2 + 3_1$, an n-colour partition of 8, than $\pi^c = 5_{5-2+1} + 3_{3-1+1} = 5_4 + 3_3$.

Definition 3.2. (see [13]) We shall call an n-colour partition π to be self-conjugate if it is identical with its conjugate π^c. Thus $5_3 + 3_2 + 1_1$ is a self-conjugate n-colour partition of 9.

Definition 3.3. (see [10]) The weighted difference of any pair of parts m_i, n_j is defined by $m - i - n - j$.

The following two theorems are n-colour analogues of Theorems 3.1 and 3.2, respectively:

Theorem 3.3. (Agarwal and Balasubramanian [13]) *Let $A(\nu)$ denote the number of n-colour partitions of a positive integer ν where the weighted difference of each pair of parts is -2. Let $B(\nu)$ denote the number of n-colour partitions of ν such that in each pair of parts $m_i, n_j (m \geq n)$ n is the arithmetic mean of the subscripts i and j. Then*

$$A(\nu) = B(\nu), \text{for all } \nu.$$

Example. $A(5) = 11$. The relevant partitions in this case are $5_1, 5_2, 5_3, 5_4, 5_5, 4_4 1_1, 3_1 2_2, 3_2 2_1, 3_3 1_1 1_1, 2_2 1_1 1_1 1_1, 1_1 1_1 1_1 1_1 1_1$. B(5) is also equal to 11, since in this case the relevant partitions are $5_1, 5_2, 5_3, 5_4, 5_5, 4_1 1_1, 3_3 2_1, 3_2 2_2, 3_1 1_1 1_1, 2_1 1_1 1_1 1_1, 1_1 1_1 1_1 1_1 1_1$.

Proof. Conjugacy is the natural bijection between the two classes. (For details see [13]). □

Theorem 3.4. (Agarwal [4]) *Let $C(\nu)$ denote the number of n-colour self-conjugate partitions of ν. Let $D(\nu)$ denote the number of symmetric plane partitions of ν. Then*

$$C(\nu) = D(\nu), \text{ for all } \nu.$$

Example. $C(5) = 4$, since there are 4 n-colour self-conjugate partitions of 5, viz., $5_3, 3_2 1_1 1_1, 2_1 2_2 1_1, 1_1 1_1 1_1 1_1 1_1$.

$D(5) = 4$, since the number of symmetric plane partitions of 5 is, also, 4. They are

$$\begin{array}{llll} 5, & 31, & 111, & 21. \\ & 1 & 1 & 11 \\ & & 1 & \end{array}$$

Proof. An n-colour partition will be self-conjugate if the parts are either self-conjugate or they appear in pairs of mutually conjugate parts. It was observed in [13] that a part can be self-conjugate if it is odd. Also, the number of pairs of mutually conjugate parts corresponding to any even integer 2ν is $[\nu/2]$, where [ ] denotes the greatest integer function. These arguments together imply that

$$(3.1) \qquad 1 + \sum_{\nu=1}^{\infty} C(\nu) q^{\nu} = \prod_{n=1}^{\infty} (1 - q^{2n-1})^{-1} (1 - q^{2n})^{-[n/2]}.$$

Theorem (3.4) follows immediately once we note that the right-hand side of (3.1) also generates the number of symmetric plane partitions (cf. [16, p. 42]). □

4. *n*-Colour Rogers-Ramanujan Identities

The following two "sum-product" identities are known as Rogers-Ramanujan identities:

$$(4.1) \qquad \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty} (1 - q^{5n-1})^{-1} (1 - q^{5n-4})^{-1},$$

and

$$(4.2) \qquad \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{n=1}^{\infty} (1 - q^{5n-2})^{-1} (1 - q^{5n-3})^{-1},$$

where

$$(a; q)_n = \prod_{i=0}^{\infty} \frac{(1 - aq^i)}{(1 - aq^{n+i})}.$$

They were first discovered by Rogers in 1894. After two decades they were rediscovered by Ramanujan and Schur, independently. MacMahon [23] gave the following combinatorial interpretations of (4.1) and (4.2), respectively:

Theorem 4.1. *The number of partitions of n into parts with minimal difference* 2 *equals the number of partitions of n into parts which are congruent to* $\pm 1 \pmod 5$.

Theorem 4.2. *The number of partitions of n with minimal part* 2 *and minimal difference* 2 *equals the number of partitions of n into parts which are congruent to* $\pm 2 \pmod 5$.

The following n-colour analogues of Theorems 4.1 and 4.2 were found in [1]:

Theorem 4.3. *The number of n-colour partitions of ν such that each pair of summands has a positive weighted difference equals the number of ordinary partitions of ν into parts* $\not\equiv 0, \pm 4 \pmod{10}$.

Example. For $\nu = 6$, we have 8 relevant partitions of each kind, viz., $6_1, 6_2, 6_3, 6_4, 6_5, 6_6, 5_1 + 1_1, 5_2 + 1_1$ of the first kind and $51, 3^2, 321, 31^3, 2^3, 2^21^2, 21^4, 1^6$ of the second kind.

Theorem 4.4. *The number of n-colour partitions of ν such that each pair of parts has a non-negative weighted difference equals the number of ordinary partitions of ν into parts* $\not\equiv 0, \pm 6 \pmod{14}$.

Example. For $\nu = 6$, we have 10 relevant partitions of each kind, viz., $6_1, 6_2, 6_3, 6_4, 6_5, 6_6, 5_1+1_1, 5_2+1_1, 5_3+1_1, 4_1+2_1$, of the first kind and $51, 42, 41^2, 3^2, 321, 31^3, 2^3, 2^21^2, 21^4, 1^6$ of the second kind.

More n-colour analogues of Theorems 4.1 and 4.2 were found in [2]. For combinatorial and analytic generalizations of Theorems 4.3 and 4.4 the reader is referred to [10, 11, 12].

5. Congruence Properties of n-Colour Partitions

The following congruence properties of the partition function $p(n)$ ($p(n)$ denotes the number of partitions of n) were found by Ramanujan [26]:

$$p(5n+4) \equiv 0 \pmod 5, \tag{5.1}$$

$$p(7n+5) \equiv 0 \pmod 7, \tag{5.2}$$

and

$$p(11n+6) \equiv 0 \pmod{11}. \tag{5.3}$$

Planar analogues of (5.1)–(5.3) were found by Cheema and Gordon [18] and Gandhi [20]. Cheema and Gordon results are:

$$t_2(\nu) \equiv 0 \pmod 5, \text{ if } \nu \equiv 3 \text{ or } 4 \pmod 5 \tag{5.4}$$

and

$$t_3(3\nu+2) \equiv 0 \pmod{3}, \tag{5.5}$$

where $t_k(\nu)$ denotes the number of k-line partitions of ν (by a k-line partition we mean a plane partition with at most k rows).

The following congruences are due to Gandhi [20]:

$$t_2(2\nu) \equiv t_2(2\nu+1) \pmod{2}, \tag{5.6}$$

$$t_3(3\nu) \equiv t_3(3\nu+1) \pmod{3}, \tag{5.7}$$

$$t_4(4\nu) \equiv t_4(4\nu+1) \equiv t_4(4\nu+2) \pmod{2}, \tag{5.8}$$

$$t_4(4\nu+3) \equiv 0 \pmod{2}, \tag{5.9}$$

$$t_5(5\nu+1) \equiv t_5(5\nu+3) \pmod{5}, \tag{5.10}$$

$$t_5(5\nu+2) \equiv t_5(5\nu+4) \pmod{5}. \tag{5.11}$$

Definition 5.1. Let $P_k(\nu)$ denote the number of n-colour partitions of ν with subscripts $\leq k$.

For example, $P_2(3) = 5$, since there are 5 n-colour partitions of 3 with subscripts ≤ 2:

$$3_1, 3_2, 2_1+1_1, 2_2+1_1, 1_1+1_1+1_1.$$

Recently in [8] the following congruence properties of n-colour partitions were proved:

$$P_2(\nu) \equiv 0 \pmod{5};\ \text{if } \nu \equiv 3 \text{ or } 4 \pmod{5}, \tag{5.12}$$

$$P_2(3\nu+2) \equiv 0 \pmod{3}, \tag{5.13}$$

$$P_2(2\nu) \equiv P_2(2\nu+1) \pmod{2}, \tag{5.14}$$

$$P_3(3\nu) \equiv P_3(3\nu+1) \pmod{3}, \tag{5.15}$$

$$P_4(4\nu) \equiv P_4(4\nu+1) = P_4(4\nu+2) \pmod{2}, \tag{5.16}$$

$$P_4(4\nu+3) \equiv 0 \pmod{2}, \tag{5.17}$$

$$P_5(5\nu+1) \equiv P_5(5\nu+3) \pmod{5}, \tag{5.18}$$

$$P_5(5\nu+2) \equiv P_5(5\nu+4) \pmod{5}. \tag{5.19}$$

We see that congruences (5.12)–(5.19) are the same as (5.4)–(5.11) with $t_k(\nu)$ replaced by $P_k(\nu)$ $(2 \leq k \leq 5)$.

Here we give a combinatorial proof of

$$T_k(\nu) = P_k(\nu), \text{for all } \nu, \text{ and } k \leq \nu, \tag{5.20}$$

from [8]. Then congruences (5.12)–(5.19) are automatically proved.

First we recall some definitions from [28] and Knuth's map φ between plane partitions of ν, on the one hand and infinite matrices $a_{ij}(i, j \geq 1)$ of non negative integer entries on the other.

If there are λ_i parts in the i-th row of a plane partition π, so that, for some r

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > \lambda_{r+1} = 0$$

then we call the partition $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ of the integer $p = \lambda_1 + \lambda_2 + \cdots + \lambda_r$ the shape of π.

If the entries of π are strictly decreasing in each column, we say that π is column strict. Knuth's map φ is the 1–1 correspondence of the following [22, Th.2]

Theorem. (Knuth) *There is a one-to-one correspondence between ordered pairs* (π_1, π_2) *of column strict plane partitions of the same shape and matrices* (a_{ij}) *of non-negative integers. In this correspondence.*

(i) k appears in π_1 exactly $\sum_i a_{ik}$ times.

(ii) k appears in π_2 exactly $\sum_i a_{ki}$ times.

A different version of this theorem known as Bender and Knuth Theorem is also found in literature (cf. Bender and Knuth [16], Nijenhuis and wiff [25]).

Theorem. (Bender and Knuth) *There is a one-to-one correspondence between plane partitions of* ν, *on the one hand, and infinite matrices* $a_{ij}(i, j \geq 1)$ *of non-negative integer entries which satisfy.*

$$\sum_{r \geq 1} r \left\{ \sum_{i+j=r+1} a_{ij} \right\} = \nu,$$

on the other.

We call images $\varphi(\pi)$ of plane partitions π of ν K_ν-matrices (K for Knuth). Although, these matrices are infinite matrices, but we represent them by largest possible square matrices containing at least one non-zero entry in the last row (or last column). Thus, for example, we will represent six K_3-matrices by

$$3, \quad \begin{matrix} 1 & 0 \\ 1 & 0 \end{matrix}, \quad \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}, \quad \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix}, \quad \begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}, \quad \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{matrix}.$$

In the proof of (5.20) we will require one more definition.

Definition 5.2. $E_{i,j}$ will denote an infinite matrix whose (i, j)-th entry is 1 and the other entries are all zeros. We call $E_{i,j}$ distinct parts of a K_ν-matrix.

Proof of (5.20). Let $\Delta = a_{1,1}E_{1,1} + a_{1,2}E_{1,2} + \cdots + a_{2,1}E_{2,1} + a_{2,2}E_{2,2} + \cdots + a_{3,1}E_{3,1} + a_{3,2}E_{3,2} + \ldots$ be a K_ν-matrix where $a_{i,j}$ are non-negative integers. We map each part $E_{p,q}$ of Δ to a single part m_i of an n-colour partition of ν. The mapping Ψ is

$$\Psi : E_{p,q} \to (p+q-1)_{p'} \tag{5.21}$$

and the inverse mapping Ψ^{-1} is easily seen to be

$$\Psi^{-1} : m_i \to E_{i,m-i+1}. \tag{5.22}$$

Under this mapping we see that each K_ν-matrix uniquely corresponds to an n-colour partition of ν and vice-versa. The composite of the two mappings φ and Ψ denoted by $\Psi.\varphi$ is clearly a bijection between plane partitions of ν, on the one hand, and the n-colour partitions of ν on the other. Next, we see that under the mapping φ every plane partition π of ν corresponds to a K_ν-matrix say, Δ, via an ordered pair (π_1, π_2) of column strict plane partitions of the same shape. Stanley [28] pointed out that the number of rows of π equals the largest part of π_2.

Now suppose

$$\Delta = \sum a_{p,q}E_{p,q} \quad \text{and} \quad \Psi(\Delta) = \sum a_{p,q}\,(p+q-1)_p.$$

Under the mapping φ the largest part of π_2 is the largest p which is the largest subscript in the corresponding n-colour partition under the mapping Ψ. This proves that under the mapping $\Psi.\varphi$ the k-line partitions of ν are mapped on to the n-colour partitions of ν enumerated by $P_k(\nu)$. This completes the proof of (5.20).

6. *n*-Colour Gaussian Polynomials

The Gaussian polynomials $\begin{bmatrix} n \\ m \end{bmatrix}$ are defined by

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{cases} (q;q)_n/(q;q)_m(q;q)_{n-m} & \text{if } 0 \le m \le n \\ 0 & \text{other wise.} \end{cases} \tag{6.1}$$

They possess the following basic properties

$$\text{degree}\begin{bmatrix} r \\ k \end{bmatrix} = k(r-k), \tag{6.2}$$

$$\begin{bmatrix} r \\ o \end{bmatrix} = \begin{bmatrix} r \\ r \end{bmatrix} = 1, \tag{6.3}$$

$$\begin{bmatrix} r \\ k \end{bmatrix} = \begin{bmatrix} r \\ r-k \end{bmatrix}, \tag{6.4}$$

$$\begin{bmatrix} r \\ k \end{bmatrix} = \begin{bmatrix} r-1 \\ k \end{bmatrix} + q^{r-k} \begin{bmatrix} r-1 \\ k-1 \end{bmatrix}, \tag{6.5}$$

$$\begin{bmatrix} r \\ k \end{bmatrix} = \begin{bmatrix} r-1 \\ k-1 \end{bmatrix} + q^{k} \begin{bmatrix} r-1 \\ k \end{bmatrix}. \tag{6.6}$$

The polynomial $\begin{bmatrix} N+M \\ M \end{bmatrix}$ generates the partition function $p(N, M, n)$ which denotes the number of partitions of n into at most M parts, each $\leq N$. That is,

$$\begin{bmatrix} N+M \\ M \end{bmatrix} = \sum_{n \geq o} p(N, M, n) q^n. \tag{6.7}$$

In [9] two different n-colour analogues of $p(N, M, n)$ were defined as follows:

Definition 6.1. Let $P_1(r, k, m, \nu)$ denote the number of n-colour partitions of ν into exactly m parts such that each subscript $b_i \leq r$ and each part $a_i \leq k + b_i - 1$.

Definition 6.2. Let $P_2(r, k, m, \nu)$ denote the number of n-colour partitions of ν into exactly m parts such that each subscript $b_i \leq r$ and each part $a_i \leq k$.

These two n-colour restricted partition functions lead to two different n-colour analogues $A_1(r, k, m; q)$ and $A_2(r, k, m; q)$ of the Gaussian polynomials given by

$$A_1(r, k, m; q) = \sum_{\nu \geq o} P_1(r, k, m, \nu) q^\nu, \tag{6.8}$$

and

$$A_2(r, k, m; q) = \sum_{\nu \geq o} P_2(r, k, m, \nu) q^\nu. \tag{6.9}$$

It was proved in [9] that

$$\sum_{\nu=0}^{\infty} \sum_{m=0}^{\infty} P_1(r, k, m, \nu) z^m q^\nu = \prod_{j=1}^{k} \frac{1}{(zq^j; q)_r}, \tag{6.10}$$

and

$$\sum_{\nu=0}^{\infty} \sum_{m=0}^{\infty} P_2(r, k, m, \nu) z^m q^\nu = \prod_{\nu=1}^{k} (1 - zq^\nu)^{-\min(r,\nu)}. \tag{6.11}$$

The proof of (6.10) follows from the fact (which is established by using the map $\Psi.(\varphi)$) that $P_1(r, k, m, \nu)$ also enumerates plane partitions of ν with $\leq r$ rows, largest part $\leq k$ and with conjugate trace m (conjugate trace of a plane partition π is defined as the number of parts $n_{ij} \geq i$) and Stanley's theorem ([28], Section 2.2, p. 56). Similarly, (6.11) can be proved by first noticing that $P_2(r, \infty, m, \nu) = \lim k \to \infty P_2(r, k, m, \nu)$ equals the number of plane partitions of ν with $\leq r$ rows and with conjugate trace m and then using Stanley's formula ([28], Eq.6, p. 59).

Analogous to properties (6.2)–(6.6) of the Gaussian polynomials, properties of the polynomials $A_1(r,k,m;q)$ and $A_2(r,k,m;q)$ were found in [9] which we state here in the form of two theorems.

Theorem 6.1. *The polynomials* $A_1(r,k,m;q)$ *satisfy the following relations*

$$\text{degree}\, A_1(r,k,m;q) = m(r+k-1), \tag{6.12}$$

$$A_1(r,k,0;q) = 1, \tag{6.13}$$

$$A_1(r,k,m;q) = A_1(k,r,m;q), \tag{6.14}$$

for $1 \leq r \leq m$

$$\begin{aligned} &q^m A_1(r,k,m;q) \\ &\quad = \sum_{j=0}^{r} \begin{bmatrix} r \\ j \end{bmatrix} (-1)^j q^{j(j+1)/2} A_1(r,k+1,m-j;q), \end{aligned} \tag{6.15}$$

and

$$A_1(r,k+1,m;q) = q^m \sum_{j=0}^{m} \begin{bmatrix} r+m-j-1 \\ m-j \end{bmatrix} A_1(r,k,j;q). \tag{6.16}$$

Theorem 6.2. *The polynomials* $A_2(r,k,m;q)$ *satisfy the following relations*

$$\text{degree } A_2(r,k,m;q) = mk, \tag{6.17}$$

$$A_2(r,k,0;q) = 1, \tag{6.18}$$

$$\textit{if } r > k \textit{ then } A_2(r,k,m;q) = A_2(k,k,m;q), \tag{6.19}$$

$$\begin{aligned} &\textit{for} 1 \leq r \leq m \quad q^m A_2(r,k,m;q) \\ &\quad = \sum_{j=o}^{r} \begin{bmatrix} r \\ j \end{bmatrix} (-1)^{j(j+1)/2} A_2(r,k+1,m-j;q). \end{aligned} \tag{6.20}$$

$$A_2(r,k,m;q) = q^m \sum_{j=o}^{m} \begin{bmatrix} r+m-j-1 \\ m-j \end{bmatrix} A_2(r,k,j;q). \tag{6.21}$$

The following theorem proved in [9] clearly shows that the polynomials $A_1(r,k,m;q)$ and $A_2(r,k,m;q)$ generalize the Gaussian polynomials.

Theorem 6.3. *We have*

$$A_1(1,k,m;q) = A_2(1,k,m;q) = q^m \begin{bmatrix} k+m-1 \\ m \end{bmatrix}, \tag{6.22}$$

and

$$\sum_{s=o}^{m} A_1(1,k,s;q) = \sum_{s=o}^{m} A_2(1,k,s;q) = \begin{bmatrix} k+m \\ m \end{bmatrix}. \tag{6.23}$$

7. n-Colour Compositions

P.A. MacMahon defined a composition as an ordered partition. Analogous to this we call an ordered n-colour partition an n-colour composition.

Thus, for example, there are 21 n-colour compositions of 4, viz.,

4_1, 4_2, 4_3, 4_4,
$3_1 1_1$, $3_2 1_1$, $3_3 1_1$, $1_1 3_1$, $1_1 3_2$, $1_1 3_3$,
$2_1 2_1$, $2_1 2_2$, $2_2 2_2$, $2_2 2_1$,
$2_1 1_1 1_1$, $2_2 1_1 1_1$, $1_1 2_1 1_1$, $1_1 1_1 2_1$, $1_1 2_2 1_1$, $1_1 1_1 2_2$,
$1_1 1_1 1_1 1_1$.

We denote the number of n-colour compositions of ν by $C(\nu)$ and the number of n-colour compositions of ν into m parts by $C(m, \nu)$. It was shown in [6] that $C(m, \nu)$ is the number of solutions of the following combinatorial problem:

Problem 1. Let there be ν balls coloured with ν-colours such that the number of balls coloured with i-th colour is not less than i. Suppose that each of these ν coloured balls has to be marked with m marks. How many different markings are possible, if there is at least one ball with each mark?

Also the following result was proved in [6]:

Theorem 7.1. *Let $C(m; q)$ and $C(q)$ denote the enumerating generating functions for $C(m, \nu)$ and $C(\nu)$, respectively. Then*

$$C(m;q) = \frac{q^m}{(1-q)^{2m}}, \tag{7.1}$$

$$C(q) = \frac{q}{1-3q+q^2}, \tag{7.2}$$

$$C(m,\nu) = \binom{\nu+m-1}{2m-1}, \tag{7.3}$$

and

$$C(\nu) = F_{2\nu'} \tag{7.4}$$

where $F_{2\nu}$ is the (2ν)th Fibonacci number.

Let $C(r, k, m, \nu)$ denote the number of n-colour compositions of ν with exactly m parts, each part $\leq k$ and each subscript $\leq r$. It was shown in [6] that $C(r, k, m, \nu)$ is the number of solutions of the following combinatorial problem:

Problem 2. Let there be ν balls coloured with r colours ($r \leq \nu$) and the number of balls coloured with i-th colour is not less than i and does not exceed k. Each of these balls is marked with m marks. How many different markings are possible, if there is at least one ball of each mark?

The following relationship:

$$C(1, k, m, \nu) = c_m(k, \nu), \tag{7.5}$$

where $c_m(k, \nu)$ is the number of ordinary compositions of ν into m parts, each $\leq k$, was shown in [6]. In view of (7.5) many of the results obtained in [6] for this generalized n-colour composition function reduce to their corresponding results for classical compositions found in the literature.

Conclusion. n-Colour partitions have also been used recently in finding Rogers-Ramanujan type identities for Frobenius partitions in [3, 5] and for plane partitions in [7]. We hope that like Euler's classical partitions these new partitions which we call n-colour partitions will find many more applications in future.

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